18.152 - Introduction to PDEs, Fall 2004 Profs. Gigliola Staffilani and Andras Vasy

Partial solutions to problem set 6

Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 58:1 $u_t = ku_{xx}, u(x,0) = e^{-x}, (x > 0)$, and $u(0,t) = 0$.

The solution is, by (6) on p. 57 of Strauss,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] e^{-y} dy.$$

Now,

$$\begin{aligned}
\frac{(x+y)^2}{4kt} + y &= \frac{y^2 + (4kt+2x)y + x^2}{4kt} = \frac{y + (2kt+x))^2 - 4kt - 4k^2t}{4kt} \\
&= \frac{(y + (2kt+x))^2}{4kt} - x - kt \\
\frac{(x-y)^2}{4kt} + y &= \frac{y^2 + (4kt-2x)y + x^2}{4kt} = \frac{y + (2kt-x))^2 + 4kt - 4k^2t}{4kt} \\
&= \frac{(y + (2kt-x))^2}{4kt} + x - kt
\end{aligned}$$

So

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{[y+(2kt-x)]^2}{4kt} + x - kt} dy - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{[y+(2kt+x)]^2}{4kt} - x - kt} dy \\
&= e^{kt-x} \left[\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(y+2kt-x)^2}{4kt}} dy \right] - e^{kt+x} \left[\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(y+2kt+x)^2}{4kt}} dy \right]
\end{aligned}$$

But letting

$$p = \frac{y + 2kt - x}{\sqrt{4kt}}, q = \frac{y + 2kt + x}{\sqrt{4kt}},$$

the two bracketed terms become

$$\frac{1}{\sqrt{\pi}} \int_{\frac{2kt-x}{\sqrt{4kt}}}^\infty e^{-p^2} dp \qquad \text{and} \qquad \frac{1}{\sqrt{\pi}} \int_{\frac{2kt+x}{\sqrt{4kt}}}^\infty e^{-q^2} dq,$$

respectively. Since $\mathrm{Erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-p^2} dp$, it follows that

$$\frac{1}{\sqrt{\pi}} \int_s^\infty e^{-p^2} dp = \frac{1}{2}(1 - \mathrm{Erf}(s)),$$

since $\int_0^\infty e^{-p^2} dp = \frac{\sqrt{\pi}}{2}$. So the brackets become

$$\frac{1}{2} - \frac{1}{2}\mathrm{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right) \qquad \text{and} \qquad \frac{1}{2} - \frac{1}{2}\mathrm{Erf}\left(\frac{2kt+x}{\sqrt{4kt}}\right),$$

respectively. Thus,

$$u(x,t) = \frac{1}{2} e^{kt-x} \left(1 - \mathrm{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right)\right) - \frac{1}{2} e^{kt+x} \left(1 - \mathrm{Erf}\left(\frac{2kt+x}{\sqrt{4kt}}\right)\right).$$

Problem 58.2 $u_t = ku_{xx}, u(x,0) = 0, u(0,t) = 1, (x > 0)$.
Let $w(x,t) = u(x,t) - 1$, so w should solve

$$w_t = kw_{xx}, w(x,0) = -1, w(0,t) = 0, (x > 0).$$

But this has been essentially solved on Ex. 1 of p. 57; the solution there has +1 initial data so the solution now is the negative of the solution given there:

$$w(x,t) = -\text{Erf}\left(\frac{x}{\sqrt{4kt}}\right).$$

Thus,

$$u(x,t) = w(x,t) + 1 = 1 - \text{Erf}\left(\frac{x}{\sqrt{4kt}}\right).$$

Problem 64.1 $u_{tt} = c^2 u_{xx}, u(x,0) = \varphi(x), u_t(x,0) = \psi(x), (x > 0)$, and $u_x(0,t) = 0$. Let $\varphi_{\text{even}}, \psi_{\text{even}}$ be the even extensions of φ and ψ.

$$\varphi_{\text{even}} = \varphi(|x|) \qquad \text{and} \qquad \psi_{\text{even}} = \psi(|x|).$$

Let $\nu(x,t)$ be the solution of

$$H(x) = \begin{cases} \nu_{tt} & = & c^2\nu_{xx} \\ \nu(x,0) & = & \varphi_{\text{even}}(x) \\ \nu_t(x,0) & = & \psi_{\text{even}}(x) \end{cases}$$

where $(x,t) \in \mathbb{R} \times (0,\infty)$.
Since $\varphi_{\text{even}}, \psi_{\text{even}}$ are even, so is the solution ν (as a function of x). Indeed, $w(x,t) = \nu(x,t) - \nu(-x,t)$ satisfies

$$w_{tt} = c^2 w_{xx}; \qquad w(x,0) = \varphi_{\text{even}}(x) - \varphi_{\text{even}}(-x) = 0; \qquad w_t(x,0) = \psi_{\text{even}}(x) - \psi_{\text{even}}(-x) = 0.$$

But then by uniqueness of solutions ot the homogenous wave equation, $w(x,t) = 0$ (since the constant 0 is certainly a solution), so $\nu(x,t) = \nu(-x,t)$ for all x, t if ν is an even function of x.

But ν even in x, so (provided that ν is differentiable, i.e. if φ, ψ are nice):

$$\nu_x(0,t) = \lim_{h\to 0} \frac{\nu(h,t) - \nu(-h,t)}{2h} = \lim_{h\to 0} \frac{0}{2h} = 0,$$

so $u(x,t) = \nu(x,t)$, $x > 0, t > c$ satisfies $u_t(x,0) = \psi_{\text{even}}(x) = \psi(x), u_x(0,t) = \nu_x(0,t) = 0$, i.e. solves the Neumann problem.

Explicitly,

$$u(x,t) = \frac{1}{2}\left[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)ds.$$

(If $x > ct$, this gives $x - ct > 0$.)

$$u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds,$$

i.e. the expected solution.

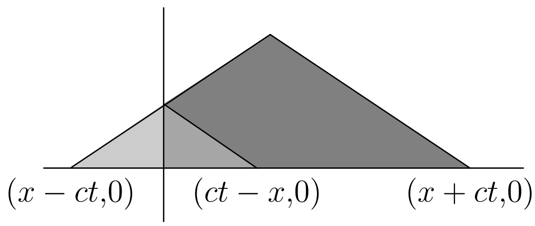
If $x < ct$, $\varphi_{\text{even}}(x-ct) = \varphi_{\text{even}}(ct-x) = \varphi(ct-x)$, and

$$\begin{aligned}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)ds &= \int_{x-ct}^{0} \psi_{\text{even}}(s)ds + \int_{0}^{x+ct} \psi_{\text{even}}(s)ds \\ &= \int_{x-ct}^{0} \psi(-s)ds + \int_{0}^{x+ct} \psi(s)ds = \int_{0}^{ct-x} \psi(-s)ds + \int_{0}^{x+ct} \psi(s)ds\end{aligned}$$

So

$$u(x,t) = \frac{1}{2}\left[\varphi(ct+x) + \varphi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(s)ds + \frac{1}{c}\int_{0}^{ct-x} \psi(s)ds.$$

Graphically, this says that the values of ψ between 0 and $ct-x$ also contribute to $u(x,t)$, unlike what happened in the Dirichlet problem.



However, note if we differentiate ν (with respect to x or t) only the values of ψ at $ct \pm x$ will be relevant, so singularities of φ, ψ still propagate along reflected characteristics!

Problem 64.3 $u(x,t) = f(x+ct)$ for $t < 0$, $x > 0$, hence this also holds up to $t = 0$ (assuming u is continuous), so $u(x,0) = f(x)$.

Also $u_t(x,t) = cf'(x+ct)$ for $t < 0, x > 0$, so we also have, in the limit, $t \to 0$.

$$u_t(x,0) = cf'(x).$$

Thus u is the solution of the Dirichlet problem.

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \qquad x > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= cf'(x) \\ u(0,t) &= 0 \end{aligned}$$

i.e. $\varphi(x) = f(x), \psi(x) = cf'(x)$. We can simply substitute into Eq. (3) on p. 60 to obtain the solution for $0 < x < ct$, and into (2) on p. 59 for $x > ct > 0$.

That gives for $x > ct > 0$:

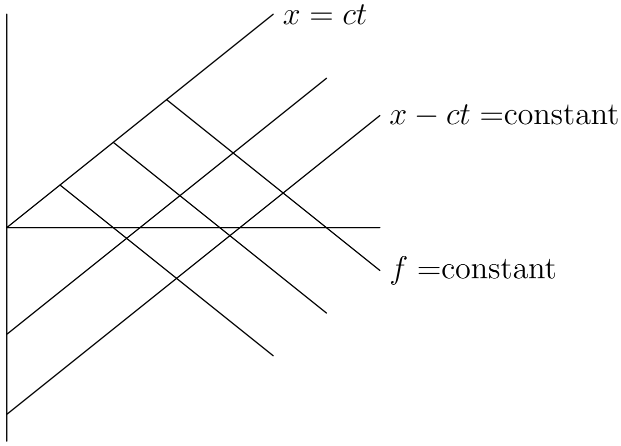
$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'(s)ds \\ &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2}\left[f(x+ct) - f(x-ct)\right] \\ &= f(x+ct) \end{aligned}$$

When we used the fundamental theorem of calculus. (Not a very surprising result!)

For $0 < x < ct$ we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(ct+x) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} cf'(s)ds \\ &= \frac{1}{2}\left[f(ct+x) - f(ct-x)\right] + \frac{1}{2}\left[f(ct+x) - f(ct-x)\right] \\ &= f(ct+x) - f(ct-x). \end{aligned}$$

Alternate solution:



Below $x = ct$ the solution must be $f(x+ct)$, since it is of the form $u(x,t) = f(x+ct) + g(x-ct)$ and for $x > 0, t < 0$, $u(x,t) = f(x+ct)$, so $g(x-ct = 0$. Since $x > 0, t < 0$ allows $x - ct$ to take any positive value, $g(s) = 0$ for $s > 0$. But that gives $g(x-ct) = 0$ if $x > ct$, i.e. $u(x,t) = f(x+ct)$ there.

To find $g(s)$ for $s < 0$, consider $u(0,t) = 0$ i.e. $f(ct) + g(-ct) = 0$. This gives $g(s) = -f(-s)$, so

$$u(x,t) = f(x+ct) - f(ct-x),$$

in agreement with the previous result.

Problem 64.5

$$\begin{aligned} u_{tt} &= 4u_{xx} \qquad x > 0 \\ u(0,t) &= 0 \\ u(x,0) &= 1 \\ u_t(x,0) &= 0. \end{aligned}$$

$\Rightarrow \varphi(x) = 1, x > 0$, and $\psi(x) = 0, x > 0$.
The solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\varphi(x+2t) + \varphi(x-2t)] \qquad x > 2t \\ u(x,t) &= \frac{1}{2}[\varphi(x+2t) - \varphi(2t-x)] \qquad 0 < x < 2t. \end{aligned}$$

So

$$u(x,t) = \begin{cases} 1, & x > 2t \\ 0, & x < 2t \end{cases}$$

Thus, the solution is singular (not even continuous) at $x = 2t$.

This is clear from the details of the reflection method as well: $\varphi_{\text{odd}}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$ is discontinuous at $x = 0$, so the solution ν will be discontinuous on the characteristic lines through $x = 0$, i.e. $x = \pm 2t$. Of these, only $x = 2t$ lies in the region $x > 0, t > 0$; this is what we found above.

Problem 64.10

$$\begin{aligned} u_{tt} &= 9u_{xx} \\ u(x,0) &= \cos x \\ u_t(x,0) &= 0. \\ u_x(0,t) &= 0 \\ u\left(\frac{\pi}{2}, t\right) &= 0. \end{aligned}$$

We thus need to extend the initial data to be even "about $x = 0$", odd "about $x = \frac{\pi}{2}$", and periodic with period $4\left(\frac{\pi}{2} - 0\right) = 2\pi$. (Note: period $= 2\pi$, not π, since conditions at the two endpoints are different.) But $\cos x$ satisfies these conditions! So the solution on the whole line is

$$\begin{aligned} \nu(x,t) &= \frac{1}{2}[\cos(x+ct) + \cos(x-ct)], \text{and so} \\ u(x,t) = \nu(x,t) &= \cos x \cos ct \qquad \text{for } 0 < x < \frac{\pi}{2}, t > 0 \\ &= \cos x \cos 3t. \end{aligned}$$