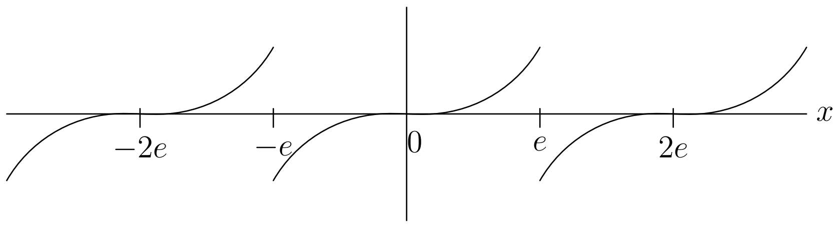
18.152 - Introduction to PDEs, Fall 2004 Profs. Gigliola Staffilani and Andras Vasy

Partial solutions to problem set 11

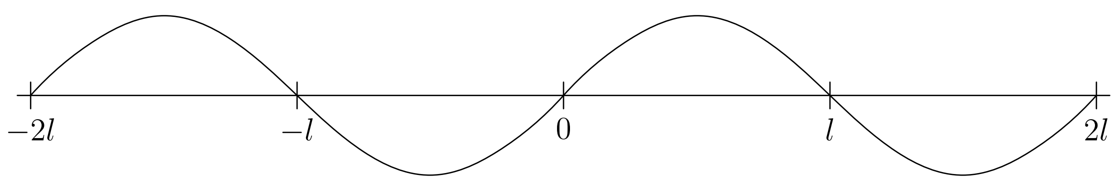
Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 129.8a) $f(x) = x^3$ on $(0, 1)$. The odd $2l-$periodic extension of f is



which is L^2, piecewise C^1, but not continuous. So exactly the same arguments as in #7 give that the Fourier sine series of f converges to f in L^2, to f pointwise on $[0, l)$, to $\frac{1}{2}[f(l+) + f(l-)] = \frac{1}{2}[l^3 + (-l)^3] = 0$ at $x = 0$, and it does not converge uniformly.

b) $f(x) = lx - x^2 = x(l - x)$. The odd $2l-$ periodic extension of f is now C^1, in fact piecewise C^2 (2nd deriv = -2 on (0,1), 2 on (-l,0)).



This suffices to give uniform convergence of the Fourier series, hence pointwise and L^2 convergence as well. To use Theorem 2 directly note that f is C^2 on $[0, l]$ and satisfies Dirichlet BC's (the BC's of the Fourier sine series), so by Theorem 2 the Fourier sine series converges to f uniformly.
c) $f(x) = x^{-2} \sin \frac{n\pi x}{l} dx$ does not even converge, since the integrand is $\geq x^{-2} \frac{n\pi x}{l} \cdot \frac{1}{2} = \frac{n\pi}{2xl}$ near $x = 0$, & $\frac{1}{x}$ is not integrable.

($\sin\theta = \theta +$ higher order terms in Taylor series, so $\sin\theta \geq \frac{\theta}{2}$ near 0).

Thus, the Fourier sine series of f makes no sense, so we cannot talk about its convergence either.

Problem 129.16: $\varphi(x) = |x|$ in $(-\pi, \pi)$. Want to approximate φ by $f(x) = \frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x$ and minimize the L^2 error $\|f - \varphi\|^2 = \int_0^\pi |f(x) - \varphi(x)|^2 dx$.

Since f is just a linear combination of the orthogonal functions 1, $\cos x, \sin x, \cos 2x, \sin 2x$, the minimizing choice is given by the Fourier coefficients (Theorem 5), i.e. if we write

$$\varphi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx),$$

then $a_0 = A_0, a_1 = A_1, a_2 = A_2, b_1 = B_1, b_2 = B_2$.

But φ is even, so the B_n are zero, hence $b_1 = b_2 = 0$, and

$$\begin{aligned}
a_0 &= A_0 = \frac{2}{\pi}\int_0^\pi x dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^\pi = \pi \\
a_n &= A_n = \frac{2}{\pi}\int_0^\pi x\cos nx dx = \frac{2}{\pi n}x\sin nx\Big|_0^\pi = -\frac{2}{n^2\pi}\cos nx\Big|_0^\pi \\
&= -\frac{2}{n\pi}\int_0^\pi \sin nx dx = -\frac{2}{n\pi}\cos nx\Big|_0^\pi = -\frac{2}{n\pi}((-1)^n - 1), n \geq 1
\end{aligned}$$

So $a_1 = -\frac{4}{\pi}$, $a_2 = 0$.