# Lecture #10

## Some important F.T in $\mathbb{R}$

| function name | $f(x)$ | $\hat{f}(\xi)$ |
|---|---|---|
| Constant | $1$ | $2\pi\,\delta_0(\xi)$ |
| Oscillation | $e^{ix\cdot a}$ | $2\pi\,\delta_a(\xi)$ |
| Gaussian | $e^{-x^2 a}$ | $\sqrt{\frac{\pi}{a}}\; e^{-\frac{\xi^2}{4a}}$ |
| Delta function | $\delta_a(x)$ | $e^{i\xi\cdot a}$ |
| Exponential | $e^{-a\lvert x\rvert}$ | $\frac{2a}{a^2+\xi^2}$ |
| Heaviside | $H(x)$ | $\pi\,\delta_0(\xi) + \frac{1}{i\xi}$ |
| Sign | $H(x)-H(-x)$ | $\frac{2}{i\xi}$ |

## Properties of F.T

| | | |
|---|---|---|
| i) | $\frac{d}{dx} f(x)$ | $i\xi\,\hat{f}(\xi)$ |
| ii) | $x\,f(x)$ | $i\frac{d}{d\xi}\hat{f}(\xi)$ |
| iii) | $f(x-a)$ | $e^{-ia\xi}\,\hat{f}(\xi)$ |
| iv) | $e^{iax}\,f(x)$ | $\hat{f}(\xi-a)$ |
| v) | $a\,f(x)+b\,g(x)$ | $a\hat{f}(\xi)+b\hat{f}(\xi)$ |
| vi) | $f(ax)$ | $\frac{1}{a}\hat{f}(\frac{\xi}{a})$ |

Proposition:

i) $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$

ii) $\mathcal{F} \circ \mathcal{F}^{-1} = Id = \mathcal{F}^{-1} \circ \mathcal{F}$

~~Propo~~: Proof of i):

We need to show that for any $f \in \mathcal{S} \Rightarrow$ for any $\alpha, \beta$

$$\left| \xi^{\alpha} \partial_{\xi}^{\beta} \hat{f}(\xi) \right| \leq C_{\alpha,\beta} \quad \text{for any } \xi.$$

From the properties

$$\partial_{\xi}^{\beta} \hat{f}(\xi) = c_0 \mathcal{F}\left( x^{\beta} f \right) \qquad ii)$$

$$\xi^{\alpha} \partial_{\xi}^{\beta} \hat{f}(\xi) = c_1 \mathcal{F}\left( \partial_x^{\alpha} (x^{\beta} f) \right) \qquad i)$$

So

$$\left| \xi^{\alpha} \partial_{\xi}^{\beta} \hat{f}(\xi) \right| = c_0 \left| \int e^{-ix\cdot\xi} \partial_x^{\alpha} \left( x^{\beta} f(x) \right) dx \right|$$

For simplicity assume $n = 1 \qquad \alpha = 1 \qquad x = 2$

$$\frac{d}{dx} \left( x^2 f(x) \right) = 2x f(x) + x^2 \frac{df}{dx}(x)$$

$$\leq C_1 \int \underset{\substack{\| \\ 1}}{\left| e^{-ix\xi} \right|} \left( \left| 2x f(x) \right| + \left| x^2 \frac{d}{dx} f(x) \right| \right) dx$$

But now we know that because $f$ is in $\mathcal{S}$

$$|x f(x)| \leq \frac{c_2}{(1+|x|)^2}$$

$$\left|x^2 \frac{d}{dx} f(x)\right| \leq \frac{c_2}{(1+|x|)^2} \qquad \text{for some } c_2 > 0$$

So we continue with

$$\leq C_3 \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^2} = C_4$$

The proof of ii) is more complex and it requires more steps.

Step 1: We already proved that if

$$\phi(x) = e^{-a|x|^2} \qquad \text{then}$$

$$\mathcal{F}\phi(\xi) = \hat{\phi}(\xi) = \left(\frac{\pi}{a}\right)^{\frac{n}{2}} e^{-|\xi|^2/4a} = \psi(\xi)$$

A similar calculation then also shows that

$$(\mathcal{F}^{-1}\psi)(x) = \phi(x)$$

In this way we proved that

$$T = \mathcal{F}^{-1} \circ \mathcal{F} \left(e^{-a|x|^2}\right) = e^{-a|x|^2}$$

Step 2:
~~[illegible]~~: if $\phi(y)=0$ and $\phi$ in $\mathcal{S}$, then $T\phi(y)=0$.
To see this we ~~write~~ only work on $\mathbb{R}$ and we write

$$\phi(x) = \phi_1(x)\,(x-y)$$

this is the Taylor expansion:

$$\phi(x)-\phi(y) = \int_0^1 \frac{d}{dt}\,\phi(tx+(1-t)y)\,dt$$

$$= \underbrace{\int_0^1 \phi'(tx+(1-t)y)\,dt}_{\substack{\parallel \\ \phi_1(x)}}\,(x-y)$$

Here we use the fact that $x_j T = T x_j$

then

$$T\phi = T(\phi_1(x)\,(x-y)) =$$

$$= T(x\phi_1(x) - y\phi_1(x)) \quad \text{linearity}$$

$$= T(x\phi_1(x)) - y\,T(\phi_1(x)) \quad \text{com. with } x$$

$$= x\,T\phi_1(x) - y\,T\phi_1(x) = (x-y)\,T\phi_1(x)$$

So $T\phi(y) = (y-y)\,T\phi_1(y) = 0$

~~Step 3~~:
~~Fact 2: there exists $e:\mathbb{R}\to\mathbb{C}$ s.t. for any $y$ $Tf(y) = e(y)f(y)$~~
~~[illegible]: Fix $y$ in $\mathbb{R}$ and take for example~~

~~$g(x) = e^{-|x-y|^2} \Rightarrow g(y) = 1$~~

Step 3: there exists a function $c: \mathbb{R} \to \mathbb{C}$ s.t. for any $y$ $Tf(y) = c(y) f(y)$ for any $f$ in $\mathcal{S}$

To prove this let $g \in \mathcal{S}$ s.t. $g(y) = 1$ (ex $g(x) = e^{-|x-y|}$) and let $c(y) = Tg(y)$. For any $f$ in $\mathcal{S}$ let

$$\phi(x) = f(x) - f(y) g(x)$$

so

$$\phi(y) = f(y) - f(y) g(y) = 0$$

thus from Step 2

$$0 = T\phi(y) = Tf(y) - f(y) Tg(y) = Tf(y) - c(y) f(y).$$

Step 4: $c(y) \equiv 1$

let $f(y) = e^{-|y|^2} \neq 0$ for all $y$ so

$$c(y) = \frac{(Tf)(y)}{f(y)}$$

But by Step 1 $(Tf)(y) = f(y)$ hence $c(y) \equiv 1$.

Q.E.D.

Convolution of two functions:

We denote with

$$f * g(x) = \int f(x-y)\, g(y)\, dy$$

<u>Fact</u>: Assume $f, g$ are in the $\mathcal{S}$, then

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$$

Proof:

$$\widehat{f * g}(\xi) = \int e^{-ix\cdot\xi} f * g(x)\, dx =$$

$$= \int e^{-ix\cdot\xi} \int f(x-y)\, g(y)\, dy\, dx =$$

$$= \iint e^{-ix\cdot\xi} f(x-y)\, g(y)\, dx\, dy$$

change variable $x - y = z$ $\quad dz\, dy = dx\, dy$

$y = y$

$$= \iint e^{-i(z+y)\cdot\xi} f(z)\, g(y)\, dz\, dy$$

$$= \iint e^{-iz\cdot\xi} e^{-iy\cdot\xi} f(z)\, g(y)\, dz\, dy$$

$$= \left(\int e^{-iz\xi} f(z)\, dz\right) \cdot \left(\int e^{-iy\cdot\xi} g(y)\, dy\right) = \hat{f}(\xi)\, \hat{g}(\xi)$$

Remark: We already have an example of convolution.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u|_{t=0} = 0 \quad u_t|_{t=0} = \phi(x) \end{cases}$$

Then $u(x,t) = S(t) * \phi(x)$

where $S(x,t) = \begin{cases} \frac{1}{2c} & |x| < ct \\ 0 & |x| > ct. \end{cases}$

(V) add Plancherel first

The Heisenberg Uncertainty Principle

In quantum mechanics

$$x = \text{position}$$

$$\xi = \text{momentum}$$

$f(x)$ = wave function

Assumption: $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$

Expected value of position $= \bar{x} = \left( \int_{-\infty}^{\infty} |x f(x)|^2 dx \right)^{\frac{1}{2}}$

" " " momentum $= \bar{\xi} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} |\xi \hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$

The uncertainty principle says

$$\bar{x} \cdot \bar{\xi} \geq \frac{1}{2}$$

Interpretation:

If we want to measure with great precision the position ( $\bar{x}$ very small ) we cannot measure at the same time with great precision also the momentum!

Say this first

Proof: Assume that $f$ is in $\mathcal{S}$

$$\left|\int_{-\infty}^{\infty} x f(x) f'(x)\, dx\right| \leq$$

$$\leq \left(\int_{-\infty}^{\infty} |x f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |f'(x)|^2 dx\right)^{\frac{1}{2}}$$

Here we used the Schwarz inequality

$$\int_{-\infty}^{\infty} |f(x)|\,|g(x)|\,dx \; \cancel{\left|\int f(x) g(x)\, dx\right|} \leq \left(\int_{-\infty}^{\infty} f^2(x)\, dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} g^2(x)\, dx\right)^{\frac{1}{2}}$$

Now by Plancherel's theorem

$$\left(\int_{-\infty}^{\infty} |f'(x)|^2 dx\right)^{\frac{1}{2}} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f'}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\xi|^2 |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} = \bar{\xi}$$

So RHS $= \bar{x}\cdot\bar{\xi}$

On the other hand

$$\int_{-\infty}^{\infty} x f(x) f'(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} x \left(f^2(x)\right)' dx$$

$$= \frac{1}{2} \underbrace{x f^2(x) \Big|_{-\infty}^{\infty}}_{=0} - \frac{1}{2}\int_{-\infty}^{\infty} f^2(x)\,dx = -\frac{1}{2}$$

hence

$$\text{LHS} = \frac{1}{2} \quad \Longrightarrow \quad \boxed{\frac{1}{2} \leq \bar{x} \cdot \bar{\xi}}$$

(*) Plancherel's theorem

b) $$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int |\hat{f}(\xi)|^2 d\xi$$

Or more in general

a) $$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \frac{1}{2\pi}\int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$$

Proof:

a) $f(x) = \mathcal{F}^{-1}\mathcal{F} f$

$$\text{LHS} = \int_{-\infty}^{\infty} \mathcal{F}^{-1}\mathcal{F} f(x)\overline{g(x)}\,dx$$

Now recall that

$$\int \hat{\phi}(\xi)\,\psi(\xi)\,d\xi = \int \phi(x)\,\hat{\psi}(x)\,dx$$

and similarly

$$\int \check{\phi}(\xi)\,\psi(\xi)\,d\xi = \int \phi(x)\,\check{\psi}(x)\,dx$$

So

$$= \int_{-\infty}^{\infty} \hat{f}(\xi)\,\mathcal{F}^{-1}\bar{g}(\xi)\,d\xi$$

$$\mathcal{F}^{-1}\bar{g}(\xi) = \frac{1}{2\pi}\int e^{ix\cdot\xi}\,\bar{g}(x)\,dx$$

$$= \frac{1}{2\pi}\overline{\int e^{-ix\cdot\xi}\,g(x)\,dx}$$

$$= \frac{1}{2\pi}\,\overline{\hat{g}}(\xi)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\,\overline{\hat{g}}(\xi)\,d\xi \qquad \text{Q.E.D.}$$