Lecture # 15

Last lecture we considered problems

$$(W)\begin{cases} u_{tt}-c^2u_{xx}=0 & 0<x<l \\ u(0,t)=u(l,t)=0 \\ u(x,0)=\phi(x) \\ u_t(x,0)=\psi(x) \end{cases} \qquad (D)\begin{cases} u_t=ku_{xx} \\ u(0,t)=u(l,t)=0 \\ u(x,0)=\phi(x) \end{cases}$$

Dirichlet Boundary

~~For this problem~~ We looked for solutions of type

$$u(x,t)=X(x)\,T(t)$$

and the Dirichlet conditions translated into solving the eigenvalue problem

$$(EP)\begin{cases} X''=-\lambda X \\ X(0)=X(l)=0 \end{cases}$$

We proved that the only eigenvalues of λ for this problem are ~~λ~~ ~~with the eigenfunctions~~

$$\lambda=\frac{n^2\pi^2}{l^2} \qquad n=1,\ldots$$

with eigenfunctions

$$X_n(x)=\sin\left(\frac{n\pi x}{l}\right)$$

Also we found that

$$(W) \Rightarrow T'' = -c^2 \lambda T$$

$$T(t) = A_n \cos \frac{n\pi c t}{\ell} + B_n \sin \frac{n\pi c t}{\ell}$$

~~so~~

$$(D) \Rightarrow T' = -k \lambda T$$

$$T(t) = A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 k t}$$

Combining $X(x)T(t)$ and linear combination one obtains the solution for the Dirichlet problem:

$$(W) \Rightarrow u(x,t) = \sum_n \left(A_n \cos \frac{n\pi c t}{\ell} + B_n \sin \frac{n\pi c t}{\ell}\right) \sin \frac{n\pi x}{\ell}$$

$$(D) \Rightarrow u(x,t) = \sum_n A_n e^{-(n\pi/\ell)^2 k t} \sin \frac{n\pi x}{\ell}$$

For the Initial data one has to impose conditions. For example

$$(D) \Rightarrow u(x,0) = \sum_n A_n \sin \frac{n\pi x}{\ell} = \phi(x)$$

We did a similar calculations for the Neumann condition. The Eigenvalue problem becomes

$$\begin{cases} X'' = -\lambda X \\ X'(0) = X'(\ell) = 0 \end{cases}$$

In this case

eigenvalues $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \quad n = 0, 1, 2 \ldots$

eigenfunctions $X_n(x) = \cos \frac{n\pi x}{\ell} \quad n = 0, 1, 2 \ldots$

Mixed Boundary Conditions

$u(0,t) = u_x(0,\ell) = 0$ mixed Dirichlet-Neumann

$$(EP) \begin{cases} -X'' = \lambda X \\ X(0) = X'(\ell) = 0 \end{cases}$$

Exercise: You will prove that

$$\lambda_n = (n+\tfrac{1}{2})^2 \frac{\pi^2}{\ell^2}$$

$$X_n(x) = \sin (n+\tfrac{1}{2})\frac{\pi x}{\ell} \quad n = 0, 1, \ldots$$

The Schrödinger Equation:

$$\begin{cases} u_t = i u_{xx} & 0<x<\ell \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

$u(x,t) = X(x)T(t)$

$T_t X = i T X_{xx} \qquad \frac{T_t}{iT} = \frac{X_{xx}}{X} = -\lambda$

$$\lambda_n = \left(\frac{n^2\pi^2}{\ell^2}\right)$$

$$(EP)\begin{cases} X'' = -\lambda X \\ X(0) = X(\ell) = 0 \end{cases} \Rightarrow X_n(x) = \sin\left(\frac{n\pi}{\ell}x\right)$$

$$n = 1, 2, \ldots$$

$$\begin{cases} T_t = -i\lambda T \\ \end{cases}$$

$$T(t) = A e^{-i\lambda t} \quad \cancel{\Rightarrow A_n e}$$

$$T_n(t) = A_n e^{-i\left(\frac{n\pi}{\ell}\right)^2 t}$$

$$u(x,t) = \sum_n A_n e^{-i\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi}{\ell}x\right)$$

So the initial condition requires the sin expansion

$$\phi(x) = \sum_n A_n \sin\left(\frac{n\pi}{\ell}x\right)$$

The Robin Condition

Recall that Robin conditions are

$$u_x(0,t) - a_0 u(0,t) = 0$$

$$u_x(\ell,t) + a_\ell u(\ell,t) = 0$$

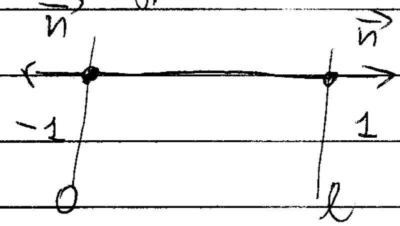
for given constants a_ℓ and a_0

This translates in the following

Eigenvalue problem

$$(EP) = \begin{cases} X'' = -\lambda X \\ X'(0) - a_0 X(0) = 0 \qquad X'(0) = a_0 X(0) \\ X'(\ell) + a_\ell X(\ell) = 0 \qquad X'(\ell) = -a_\ell X(\ell) \end{cases}$$

Why the different signs of a_0 and a_ℓ



$a_0 = a_\ell = 0 \implies$ insulation (no heat escapes)

$a_0, a_\ell > 0 \implies$ radiation

$a_0, a_\ell < 0 \implies$ absorption

$\boxed{\text{Solve } EP \text{ with } \lambda = \beta^2 > 0, \ \beta > 0}$

The general solution for the first equation in (EP) is

$$X(x) = C\cos\beta x + D\sin\beta x$$

$$X'(x) = -\beta C\sin\beta x + D\beta\cos\beta x$$

$$X'(x) \pm a X(x) = (\pm aD - \beta C)\sin\beta x + (D\beta \pm aC)\cos\beta x$$

$x=0$

$$0 = (\pm a D - \beta c)\cdot 0 + (D\beta \pm a C)$$

$$D\beta - a_0 C = 0 \qquad D = \frac{a_0 C}{\beta}$$

$x = \ell$

$$0 = (a_\ell D - \beta c)\sin\beta\ell + (D\beta + a_\ell C)\cos\beta\ell$$

$$0 = \left(a_\ell a_0 \frac{C}{\beta} - \beta c\right)\sin\beta\ell + (a_0 C + a_\ell C)\cos\beta\ell$$

$$C\left[\left(\frac{a_\ell a_0}{\beta} - \beta\right)\sin\beta\ell + (a_0 + a_\ell)\cos\beta\ell\right] = 0$$

$C \neq 0$ because we don't want the trivial solution

$$(a_\ell a_0 - \beta^2)\sin\beta\ell + \beta(a_0 + a_\ell)\cos\beta\ell = 0$$

If $\cos\beta\ell = 0$ then from above

$a_\ell a_0 = \beta^2 = \lambda$ only possible if $a_\ell a_0 > 0$

If $\cos\beta\ell \neq 0$

$$\boxed{(\beta^2 - a_\ell a_0)\tan\beta\ell = \beta(a_0 + a_\ell)} \quad (\text{A})$$

One needs to solve for β to get λ.

Once β is found the eigenfunctions are

$$X(x) = C\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right)$$

By previous discussion

$$\beta^2 - a_e a_0 \neq 0$$

So $\tan\beta\ell = \dfrac{\beta(a_0+a_e)}{\beta^2 - a_e a_0}$

Because we cannot solve (*) directly we will only approximate the values of β solving (*).

Consider the two functions as functions of β

$$y = \tan\beta\ell$$

$$y = \beta\frac{(a_0+a_e)}{\beta^2 - a_e a_0}$$

We look at their graphs and in particular at the point of intersection.

<u>Case 1</u>: Radiation at both ends $(a_0, a_e > 0)$

We graph first $y = \underbrace{\beta\dfrac{(a_0+a_e)}{\beta^2 - a_e a_0}}_{f(\beta)}$ $\beta > 0$

8

$$\lim_{\beta \to +\infty} f(\beta) = 0$$

$$\lim_{\beta \to \sqrt{a_e a_o}^+} f(\beta) = +\infty$$

$$\lim_{\beta \to \sqrt{a_e a_o}^-} f(\beta) = -\infty$$

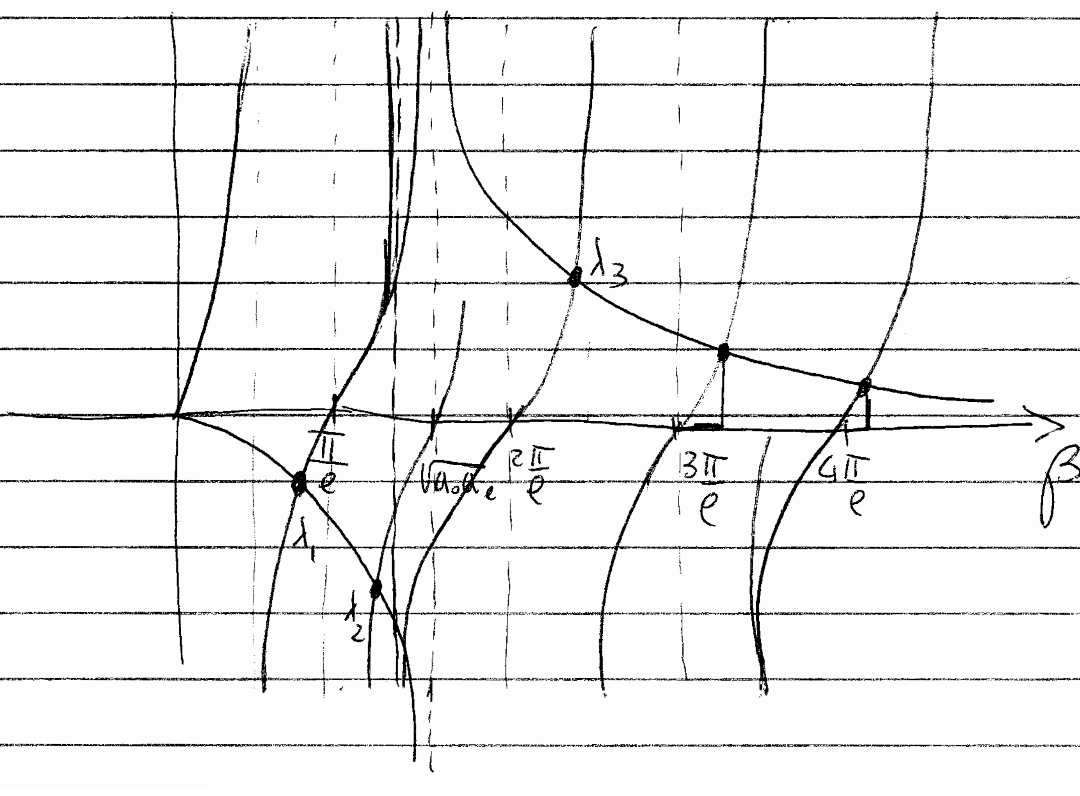
$a_o + a_e = a > 0$
$a_e a_o = b > 0$

$$f'(\beta) = \frac{(a_o + a_e)}{\beta^2 - a_e a_o} - \beta \frac{(a_o + a_e) 2\beta}{(\beta^2 - a_e a_o)^2}$$

$$= \frac{a\beta^2 - a a_e a_o - 2\beta^2 a}{(\beta^2 - a_e a_o)^2}$$

$$= -\frac{a(a_e a_o + \beta^2)}{(\quad)^2} < 0$$

$f(0) = 0$



From the graph we have that

$$n\frac{\pi}{\ell} < \beta_n < (n+1)\frac{\pi}{\ell}$$

$$\left(\frac{n\pi}{\ell}\right)^2 < \lambda_n < (n+1)^2\left(\frac{\pi}{\ell}\right)^2 \qquad n = 0, 1, 2 \ldots$$

$$\lim_{n\to\infty}\left(\beta_n - \frac{n\pi}{\ell}\right) = 0$$

Case 2: absorption at $x=0$ $\quad a_0 < 0$
radiation at $x=\ell$ $\quad a_\ell > 0 \Rightarrow a_0 a_\ell < 0$
we also assume that $\quad a_0 + a_\ell > 0$
more radiation than absorption

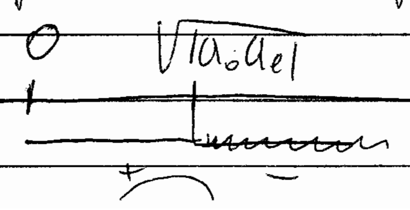
Set $a_0 a_\ell = -b^2$ and $a_0 + a_\ell = c^2$

$$y = \beta\frac{c^2}{\beta^2 + b^2}$$

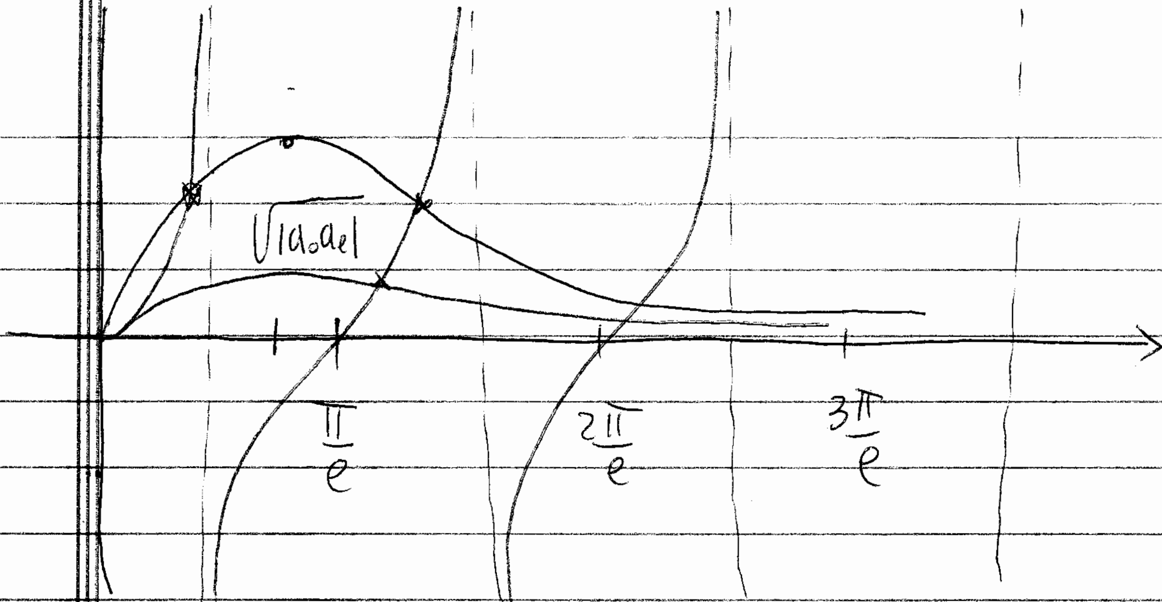
So the function is non singular

$$y' = \frac{c^2}{\beta^2+b^2} - \beta\frac{c^2\, 2\beta}{(\beta^2+b^2)^2} = \frac{c^2\beta^2 + c^2 b^2 - 2\beta^2 c^2}{(\beta^2+b^2)^2}$$

$$= \frac{c^2(b^2-\beta^2)}{(\beta^2+b^2)^2} = \frac{c^2}{(\beta^2+b^2)^2}(b-\beta)(b+\beta)$$



$b = \sqrt{|a_0 a_\ell|}$



$$\lim_{\beta \to +\infty} f(y) = 0$$

Wether there is an eigenvalue in the interval $[0, \frac{\pi}{e}]$ depends on the initial slope of the function $f(\beta)$. We will discuss this in a ~~p~~ moment.

Otherwise it is still true that

$$\frac{n^2\pi^2}{e^2} < \lambda_n < (n+1)^2 \frac{\pi^2}{e^2} \quad \text{for } n \geq 1$$

<u>Discussion for λ_0:</u>

$\exists \lambda_0 \iff$ slope of $f(\beta)\big|_{\beta=0} >$ slope $\tan \beta e\big|_{\beta=0}$

$$f'(\beta) = \frac{c^2 b^2}{(\cancel{\beta^2} + b^2)^2} = \frac{(a_0 + a_e)}{b^2} = \frac{a_0 + a_e}{-a_0 a_e}$$

$$(\tan \beta e)'\big|_{\beta=0} = e \sec^2(\beta e)\big|_{\beta=0} = e$$

So the condition is

$$\frac{a_0+a_\ell}{-a_0 a_\ell} > \ell$$

$$\iff \boxed{-a_0 a_\ell \ell < a_0+a_\ell} \qquad (**)$$

So λ_0 exists $\iff$ $(*)$.

$\boxed{\text{Zero eigenvalues}}$

We go back to (EP)

$$\begin{cases} X''=0 & X(x)=Ax+B \\ X'(0)-a_0X(0)=0 & \\ X'(\ell)+a_\ell X(\ell)=0 & \end{cases}$$

$$X'(x)=A \qquad A-a_0B=0$$

$$A+a_\ell.(A\ell+B)=0$$

$$a_0B+a_\ell(+\ell a_0B+B)=0$$

$$B\neq 0 \qquad +a_0+a_\ell(+\ell a_0+1)=0$$

$$+a_0+\ell a_\ell a_0+a_\ell=0$$

$$a_0+a_\ell=-\ell a_\ell a_0$$

So 0 is eigenvalue $\iff$ $a_0 + a_\ell = -\ell a_\ell a_0$

In particular a_0, a_ℓ have to have opposite signs!

$\boxed{\text{Negative eigenvalues}}$

We now assume $\lambda = -\gamma^2 < 0$, $\gamma > 0$

If one repeats the argument γ is the solution of

$$\tanh \gamma \ell = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a_\ell}$$

with eigenfunctions

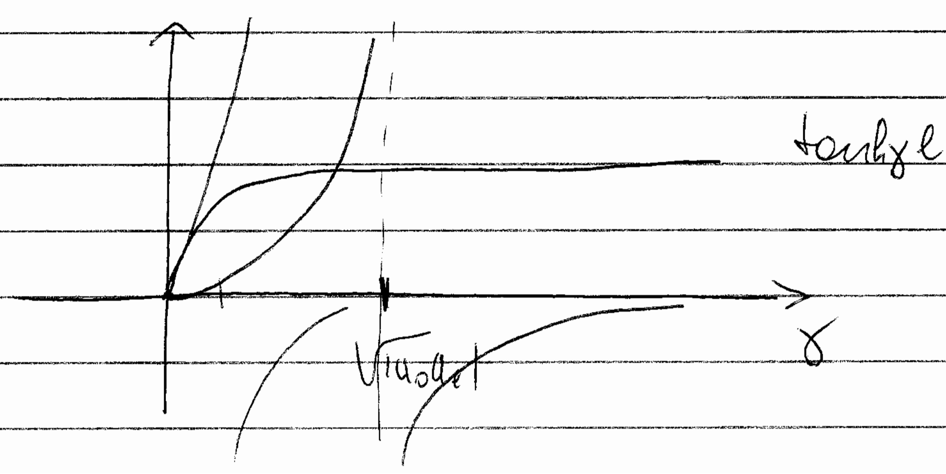
$$X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x$$

<u>Case 1</u>: $a_0, a_1 > 0$

Observe that for $\gamma > 0$ $\tanh \gamma > 0$

So this cannot happen $\Rightarrow$ no negative eigenvalues

<u>Case 2</u>: $a_0 < 0$, $a_\ell > 0$ $a_0 + a_\ell > 0$



$$\tanh z = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

$$\lim_{z \to +\infty} \tanh z = 1$$

$$\tanh 0 = 0$$

$$f(\gamma) = -\frac{(a_0 + a_e)\gamma}{\gamma^2 + a_0 a_e} \qquad \gamma^2 = -a_0 a_e$$

$$\gamma = \sqrt{|a_0 a_e|}$$

$$\lim_{\gamma \to +\infty} f(\gamma) = 0$$

$$\lim_{\gamma \to \sqrt{|a_0 a_e|}^{\pm}} f(\gamma) = \pm\infty \qquad f(0) = 0$$

Either there is 1 eigenvalue λ_0 or none.
It depends on the slope of $f(\gamma)$.

$$(*) \qquad -\frac{(a_0 + a_e)}{a_0 a_e} < \ell$$

$\downarrow$

slope f at 0 slope $\tanh \gamma \ell$ at zero

$(*) \iff \exists$ one eigenvalue λ_0