Lecture #21

So for all x in $[-\pi,\pi]$

$$|f(x)-S_N(x)| \le \sum_{n=N+1}^{\infty} |A_n \cos nx + B_n \sin nx|$$

$$\le \sum_{n=N+1}^{\infty} |A_n|+|B_n| \xrightarrow[N\to\infty]{} 0$$

by absolute convergence above

So

$$\max_{-\pi\le x\le\pi} |f(x)-S_N(x)| \xrightarrow[N\to\infty]{} 0 \quad \text{uniform convergence.}$$

To prove Theorem 3 on L^2-convergence we need the following

Theorem: the set of functions ~~with~~ on $[a,b]$ that have continuous first derivatives ($=C^1([a,b])$) is dense in $L^2(a,b)$, that is: For any ~~ε>0 there is f in~~ f in $L^2(a,b)$ and for any $\varepsilon>0$ there exists h_ε in $C^1[a,b]$ s.t.

$$\|f-h_\varepsilon\|_{L^2} < \varepsilon$$

Assume this theorem.

<u>Proof of theorem</u> 3 (L^2 convergence): Fix $\varepsilon > 0$

Let $\sum_{n=0}^{N} A_n X_n$ and $\sum_{n=0}^{N} A_n^\varepsilon X_n$ the partial sum of the classical F. Series of f and h^ε respectively. Then

$$\| f(x) - \sum_{n=0}^{N} A_n X_n \|_{L^2} =$$

$$\| f(x) - h^\varepsilon(x) + h^\varepsilon(x) - \sum_{n=0}^{N} A_n^\varepsilon X_n(x) + \sum_{n=0}^{N} A_n^\varepsilon X_n(x) - \sum_{n=0}^{N} A_n X_n(x) \|_{L^2}$$

$$\leq \| f(x) - h^\varepsilon(x) \|_{L^2} + \| h^\varepsilon(x) - \sum_{n=0}^{N} A_n^\varepsilon X_n(x) \|_{L^2}$$

$$+ \| \sum_{n=0}^{N} (A_n^\varepsilon - A_n) X_n(x) \|_{L^2}$$

Now observe that

$$\left(\int \sum_{n=0}^{N} (A_n^\varepsilon - A_n)^2 X_n^2(x)\, dx \right)^{\frac{1}{2}} =$$

$$= \left(\sum_{n=0}^{N} |A_n^\varepsilon - A_n|^2 \int X_n^2(x)\, dx \right)^{\frac{1}{2}}$$

Because $A_n^\varepsilon - A_n = \int_a^b \left(h^\varepsilon(x) - f(x)\right) X_n(x)\, dx \, \frac{1}{\|X_n\|_{L^2}^2}$

It follows by Bessel's inequality

$$\left\| \sum_{n=0}^{N} (A_n^\varepsilon - A_n) X_n(x) \right\|_{L^2} \leq \| h^\varepsilon - f \|_{L^2} \leq \varepsilon$$

So

$$\left\| f(x) - \sum_{n=0}^{N} A_n X_n \right\|_{L^2} \leq \varepsilon + \left\| h^\varepsilon(x) - \sum_{n=0}^{N} A_n^\varepsilon X_n(x) \right\|_{L^2} + \varepsilon \quad 3\varepsilon$$

But now because the uniform convergence

$$\| h^\varepsilon - \quad \|_{L^2} \leq \max_{(a,b)} \left| h^\varepsilon(x) - \sum_{n=0}^{\infty} A_n^\varepsilon X_n(x) \right| (b-a)$$

$$\downarrow N \to \infty$$

$$0$$

So given $\varepsilon > 0$ there exists $N_\varepsilon > 0$ s.t. if $N \geq N_\varepsilon$

$$\left\| f(x) - \sum_{n=0}^{N} A_n X_n \right\|_{L^2} \leq 3\varepsilon \qquad \text{Q.E.D.}$$

Proof of Pointwise convergence

Theorem 4': If $f(x)$ is a function of period 2ℓ, $f(x)$, $f'(x)$ piecewise continuous, then the classical Fourier Series

$$(*) \quad \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{\ell} + B_n \sin\frac{n\pi}{\ell}x$$

converges to $\frac{1}{2}\left[f(x+)+f(x-)\right]$ pointwise

Proof: w.l.o.g. set $\ell=\pi$. Let's first start by assuming f in $C^1[-\pi,\pi]$ (f' continuous) then $(*)$ can be written as

$$(*)' \quad \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

$$A_n = \int_{-\pi}^{\pi} f(y)\cos yn \,\frac{dy}{\pi} \qquad n=0,1,\ldots$$

$$B_n = \int_{-\pi}^{\pi} f(y)\sin ny \,\frac{dy}{\pi} \qquad n=1,\ldots$$

We want to prove that if

$$S_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos nx + B_n \sin nx$$

then $\lim_{N\to\infty} S_N(x) = f(x)$ for any fixed x in $[-\pi,\pi]$

Notice that

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) + \left(2\sum_{n=1}^{N} \cos ny \cos nx + \sin ny \sin nx\right) f(y)\, dy$$

by trigonometric identity:

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\cos ny \cos nx + \sin ny \sin nx = \cos(n(y - x))$$

so if we set $K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta$, then

↳ <u>Dirichlet Kernel</u>

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y-x)\, f(y)\, dy$$

<u>Properties of Dirichlet kernel</u>

i) $K_N(\theta)$ is a periodic function with period 2π

ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)\, d\theta = 1$ for all N

iii) $K_N(\theta) = \dfrac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$

<u>Proof:</u>

i) the periodicity comes from the linear finite combination of $\cos n\theta$ and 1, all of them periodic 2π.

ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)\, d\theta = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta}_{=1} + \frac{1}{2\pi}\sum_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta \, d\theta}_{=0}$

iii) $K_N(\theta) = 1 + \sum_{n=1}^{N} (e^{in\theta} + e^{-in\theta}) = \sum_{z=-N}^{N} e^{iz\theta}$

this is a geometric series and for this kind the sums are known:

$$K_N(\theta) = \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} =$$

$$= \frac{e^{-iN\theta} - e^{iN\theta}e^{i\theta}}{1 - e^{i\theta}} = \frac{e^{-i(N+\frac{1}{2})\theta}e^{i\frac{\theta}{2}} - e^{+i(N+\frac{1}{2})\theta}e^{i\frac{\theta}{2}}}{1 - e^{i\theta}}$$

$$= \frac{e^{-i(N+\frac{1}{2})\theta} - e^{i(N+\frac{1}{2})\theta}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}} \quad \checkmark$$

Now back to $S_N(x)$: set $y - x = \theta$, then

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)\, f(x+\theta)\, d\theta$$

↳ change of variables and some periodicity functions.

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)\, f(x+\theta)\, d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)\, d\theta\, f($$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)\,[f(x+\theta) - f(x)]\,dx$$

set $\quad g(\theta) = \dfrac{f(x+\theta) - f(x)}{\sin\frac{1}{2}\theta}$

$$= \int_{-\pi}^{\pi} g(\theta)\,\sin(N+\tfrac{1}{2})\theta\,\frac{d\theta}{2\pi}$$

<u>Fact</u> : $\{\phi_N(\theta) = \sin[(N+\frac{1}{2})\theta] \quad N = 1, 2, \ldots$

is an orthogonal set on the interval $(0, \pi)$
because they are eigenfunctions with symmetric
boundary conditions (see homework!)
So from the Bessel's inequality

$$\sum_{N=1}^{\infty} \underbrace{\frac{|(g, \phi_N)|^2}{\|\phi_N\|^2}}_{\tilde{A}_N} \le \|g\|_{L^2}^2$$

Now $\quad \|g\|_{L^2}^2 = \displaystyle\int_{-\pi}^{\pi} \frac{[f(x+\theta) - f(x)]^2}{\sin^2\frac{1}{2}\theta}\,d\theta \qquad x$ fixed

$$= \int_{|\theta|<\varepsilon} + \int_{|\theta|>\varepsilon}$$

$$\int_{|\theta|<\varepsilon}$$

$\left|\sin^2 \frac{1}{2}\theta\right| \sim \frac{1}{4}\theta^2$ for $|\theta|$ small (taylor expansion)

$$\frac{[f(x+\theta) - f(x)]^2}{\sin^2 \frac{1}{2}\theta} \sim \frac{[f(x+\theta) - f(x)]^2}{\theta^2}$$

$$\sim \frac{f'(\xi_\theta)^2 (x+\theta-x)^2}{\theta^2} \simeq [f'(\xi_\theta)]^2 \le C \text{ in } [-\pi, \pi]$$

So

$\int_{|\theta|<\varepsilon} \le C_\varepsilon$ on the other hand

for $|\theta| > \varepsilon$ $\sin^2 \frac{1}{2}\theta > C_\varepsilon$

$$\int_{|\theta|>\varepsilon} \frac{[f^2(x+\theta) - f(\theta)]^2}{\sin^2 \frac{1}{2}\theta} \le \frac{2 \max f^2}{C_\varepsilon} 2\pi$$

So $\sum_{N=1}^{\infty} \frac{|(g,\phi_N)|^2}{\|\phi_N\|^2}$ converges $\Rightarrow \lim_{N\to\infty} \frac{|(g,\phi_N)|^2}{\|\phi_N\|^2} = 0$

but $\|\phi_N\|_{L^2}^2 = \pi$ so $\lim_{N\to\infty} |(g,\phi_N)| = 0$

$\Leftrightarrow$ $\lim_{N\to\infty} |S_N(x) - f(x)| = 0$ for all fixed x.

If f and f' have finitely many discontinuities then the proof is the one above for the continuous points. For the jump discontinuities one looks at

$S_N(x) - \frac{1}{2}[f(x-) + f(x+)]$ and repeats the argument.

~~The~~ ~~The Gibbs Phenomenon~~

Inhomogeneous Boundary Conditions

Diffusion eq:

$$\begin{cases} u_t = k u_{xx} \\ u(0,t) = h(t) \quad u(\ell,t) = j(t) \\ \quad u(x,0) = 0 \end{cases}$$

Clearly a separation of variable technique cannot work because the variable are mixed at the boundary.

(It is enough L^2 really!)

If we assume that $u(x,t)$ and $u_t(x,t)$ are continuous on $[0,\ell]$ for fixed t then they are in $L^2(0,\ell)$

So by completeness

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi}{\ell} x$$

$$\partial_t u(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi}{\ell} x$$

with $u_n(t) = \frac{2}{\ell} \int_0^\ell u(x,t) \sin \frac{n\pi}{\ell} x \, dx$

$$v_n(t) = \frac{2}{\ell} \int_0^\ell \frac{\partial}{\partial t} u(x,t) \sin \frac{n\pi}{\ell} x \, dx$$

$$= \frac{2}{\ell} \frac{\partial}{\partial t} \int_0^\ell u(x,t) \sin \frac{n\pi}{\ell} x \, dx$$

$$= \frac{d}{dt} u_n(t)$$

Clearly if u_t cont, from the equation

$u_t = k\, u_{xx}$ also u_{xx} is

$$u_{xx} = \sum_{n=1}^{\infty} w_n(t) \sin \frac{n\pi x}{\ell}$$

$$w_n(t) = \frac{2}{\ell} \int_0^\ell \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{\ell}\, dx$$

By integration by parts

$$\int_0^\ell \partial_{xx} u \sin \frac{n\pi x}{\ell}\, dx = \partial_x u \sin \frac{n\pi x}{\ell} \Big|_0^\ell - \int \partial_x u \frac{n\pi}{\ell} \cos\left(\frac{n\pi}{\ell} x\right)$$

$$= \underbrace{u_x \sin \frac{n\pi}{\ell} x}_{=0} - u \frac{n\pi}{\ell} \cos \frac{n\pi}{\ell} x \Big|_0^\ell - \int u \left(\frac{n\pi}{\ell}\right)^2 \sin\left(\frac{n\pi}{\ell} x\right) dx$$

$$\boxed{w_n(t) = -\left(\frac{n\pi}{\ell}\right)^2 u_n(t) - \frac{2}{\ell} \frac{n\pi}{\ell} j(t) (-1)^n + \frac{2}{\ell} \frac{n\pi}{\ell} h(t)}$$

From the equation

$$v_n(t) - k\, w_n(t) = \frac{2}{\ell} \int_0^\ell \underbrace{(u_t - k\, u_{xx})}_{=0} \sin \frac{n\pi x}{\ell}\, dx = 0$$

500

$$\begin{cases} \dfrac{d}{dt} u_n = v_n(t) = K \left[-\left(\dfrac{n\pi}{\ell}\right)^2 u_n - \dfrac{2}{\ell^2} n\pi\, j(t)(-1)^n + \dfrac{2}{\ell^2} n\pi\, h(t) \right] \\ u_n(0) = 0 \end{cases}$$

this is an O.D.E for all n

$$u_n(t) = C e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} - 2n\pi \ell^{-2} k \int_0^t e^{-\left(\frac{n\pi}{\ell}\right)^2 k(t-s)} \left[(-1)^n j(s) - h(s)\right] d$$

but imposing $u_n(0) = 0 \Rightarrow C = 0$

Similar ideas work for the wave equation.