So the series converges pointwise to 0 (8)

Uniform convergence

$$\max_{(0,1]} \frac{Nx}{1+N^2x^4}$$

$$g'(x) = \frac{N(1+N^2x^4) - Nx\,4N^2x^3}{(1+N^2x^4)^2} = \frac{N + N^3x^4 - 4N^3x^4}{(\quad)^2} = 0$$

$$3N^{\cancel{3}}x^4 = \cancel{N} \qquad x^4 = \frac{1}{3N} \qquad x = \frac{1}{\sqrt[4]{3N}}$$

$$\max_{(0,1]}(\quad) = \frac{N(3N)^{-\frac{1}{4}}}{1+N^{\cancel{2}}\frac{1}{3\cancel{N}}} = C\,\frac{N^{\frac{3}{4}}}{1+\frac{N}{3}} \xrightarrow[N\to\infty]{} 0$$

Clearly L^2 convergence follows.

Back to Fourier Series: $\boxed{\text{Lecture \# 20}}$

Theorem 2: (Uniform Convergence)

The Fourier Series $\sum A_n X_n(x)$ converges to $f(x)$ uniformly in $[a,b]$ provided

i) $f(x)$, $f'(x)$, $f''(x)$ exist and are continuous on $[a,b]$

ii) $f(x)$ satisfies the given symmetric boundary conditions.

Theorem 3: (L^2 convergence)

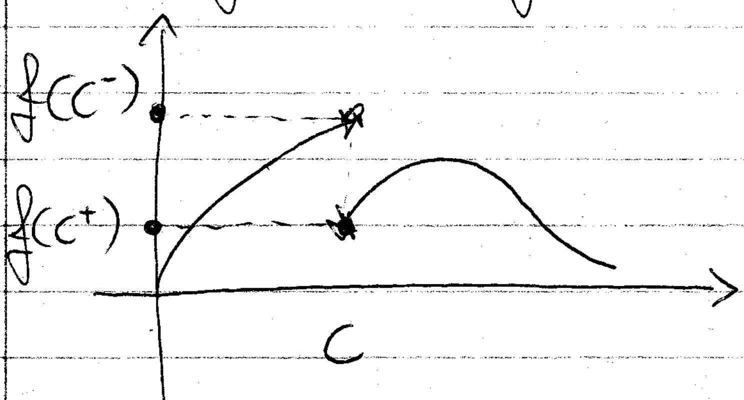
For any f in $L^2([a,b])$ ~~[illegible]~~ ~~[illegible]~~ its F.S. $\sum A_n X_n(x)$ converges to f in L^2 sense.

Definition: A function f has a jump discontinuity at a point c if

$$\lim_{x \to c^+} f(x) = f(c^+) \text{ exists}$$

$$\lim_{x \to c^-} f(x) = f(c^-) \quad " \quad "$$

but $f(c^+) \neq f(c^-)$



Remark: By this definition f may not even be defined at c!

Definition: f is piecewise continuous on $[a,b]$ if f admits only finitely many points of jump discontinuity.

Theorem 4 (Pointwise Convergence)

i) The classical Fourier Series (full, sin or cos) converges to $f(x)$ pointwise in (a,b), provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$

ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point x in $-\infty < x < \infty$, the sum is

$$\sum_n A_n X_n(x) = \frac{1}{2}\left[f(x+) + f(x-)\right]$$

for all $a < x < b$.

$$= \frac{1}{2}\left[f_{ext}(x+) + f(x-)\right]$$

for all the other x and $f_{ext}(x)$ is the extended function (periodic, odd periodic, even periodic,

Theorem 4': If $f(x)$ is a function of period 2ℓ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for all x.

Example

$f(x) = 1$ on $(0,\pi)$. This is an even function so its F.S. is sin type

$$A_n = \frac{2}{\pi}\int_0^\pi 1 \cdot \sin n\cancel{\pi}x \, dx$$

$$nx = y \quad dx = \frac{1}{n}dy \qquad = \frac{2}{n\pi}\int_0^{n\pi} \sin y \, dy = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

$$1 = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin nx$$

$\downarrow$

in what sense ?

By theorem 4 the convergence is definitely pointwise in $0 < x < \pi$. in (a,b)

Is the convergence uniformely? A uniform convergence would imply ~~a~~ pointwise convergence at 0 and at π.

But

$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin nx \Big|_{x=0,\pi} = 0 \neq 1$$

So no uniform convergence.

~~This is~~ Does this contradict theorem 2 ?

No. In fact i) is true since $f(x) = 1$ is C^∞, but (ii) is not true: the ~~Bcs~~ BC for a sin series are Dirichlet, but $f(0) = f(\pi) \neq 0$!

Remark: Because theorem 4 it follows that for any fixed x_0 in $(0,\pi)$

$$1 = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin n x_0$$

In particular $x_0 = \frac{\pi}{2} \Rightarrow \sin n\frac{\pi}{2} = 1$

$$1 = \frac{\pi}{2}\sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n} = \frac{\pi}{2}\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$$

In this way we can calculate the series of sequences as above.

Remark 2: the pointwise convergence is very weak. For example if we take derivative the convergence is not preserved in general

$$(1)' \overset{?}{=} \frac{4}{\pi}\sum_{n \text{ odd}} (\sin nx)'$$

$$0 \overset{?}{=} \frac{4}{\pi}\sum_{n \text{ odd}} n \cos nx$$

the series does not converge because $\lim_{n\to\infty} n \cos nx$ does not exist!

Lecture #20 continued

~~Theorem~~ Recall again that f in $L^2((a,b)) \iff$

$$\|f\|_{L^2} = \left(\int_a^b |f|^2(x)\,dx\right)^{\frac{1}{2}} < \infty$$

Theorem 5: Let $\{X_n\}$ be any orthogonal set of functions in $L^2(a,b)$. Let $\|f\|_{L^2} < \infty$. Let N be a fixed positive integer. Among all possible choices of N constants $c_1, \dots, c_N$, the choice that minimize

$$\left\| f - \sum_{n=1}^{N} c_n X_n \right\|_{L^2}$$

is $c_1 = A_1, \dots, c_n = A_n$ when

$$A_i = \frac{\langle f, X_i \rangle}{\langle X_i, X_i \rangle}$$

Proof: W.l.o.g. assume X_n real valued and f also.

$$E_N = \left\| f - \sum_{n \le N} c_n X_n \right\|_{L^2}^2 = \int_a^b \left| f(x) - \sum_{n=1}^{N} c_n X_n(x) \right|^2 dx$$

take the square

$$E_N = \int_a^b |f(x)|^2 dx - 2\sum_{n=1}^{N} c_n \int f(x) X_n(x)\, dx$$
$$+ \underbrace{\sum_{n=1}^{N}\sum_{m=1}^{N} c_n c_m \int X_n(x) X_m(x)\, dx}_{\displaystyle \overset{\shortparallel}{\sum_{n=1}^{N} c_n^2 \int_a^b |X_n|^2 dx}}$$

$$E_N = \|f\|_{L^2}^2 - 2\sum_{n=1}^{N} c_n \langle f, X_n\rangle + \sum_{n=1}^{N} c_n^2 \|X_n\|_{L^2}^2$$

Complete the square

$$= \|f\|_{L^2}^2 + \sum_{n=1}^{N} \|X_n\|_{L^2}^2 \left[c_n - \frac{\langle f, X_n\rangle}{\|X_n\|^2} \right]^2$$
$$- \sum_{n=1}^{N} \frac{\langle f, X_n\rangle^2}{\langle X_n, X_n\rangle}$$

Clearly the smallest E_N is obtained when

$$[\ ] = 0 \iff c_n = \frac{\langle f, X_n\rangle}{\|X_n\|^2}$$

Q.E.D.

Remark:

If $c_n = A_n$ then above we get

$$0 \le E_n = \|f\|_{L^2}^2 - \sum_{n=1}^{N} A_n^2 \langle X_n, X_n\rangle$$

so in particular

$$\sum_{n=1}^{N} A_n^2 \, \|X_n\|^2 \le \|f\|_{L^2}^2$$

$$\Updownarrow$$

$$\sum_{n=1}^{N} A_n^2 \int_a^b |X_n(x)|^2 dx \le \int_a^b |f(x)|^2 dx$$

(<u>Bessel's</u> inequality)

<u>Theorem 6</u>: Assume f is in L^2. The Fourier Series of $f(x)$ converges to $f(x)$ in L^2 iff

$$(*) \quad \sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |f(x)|^2 dx$$

(<u>Parseval's</u> equality)

<u>Proof</u>:

Using the above notation

$$\sum_{n=1}^{\infty} A_n X_n \xrightarrow{L^2} f \iff$$

$$\lim_{N\to\infty} \Big\| \sum_{n=1}^{N} A_n X_n - f \Big\|_{L^2}^2 = 0 \iff \lim_{N\to\infty} E_N = 0$$

But we just proved that

$$E_N = \int_a^b |f(x)|^2 dx - \sum_{n=1}^{N} A_n^2 \int_a^b |X_n(x)|^2 dx$$

So $(*)$.

<u>Definition</u>: An infinite orthogonal set of functions in $L^2(a,b)$
$\{X_1(x), X_2(x), \ldots\}$ is called <u>complete</u> if
the <u>Parseval equality holds</u>. ~~(⟺ Any~~ for any
function f in $L^2(a,b)$. (⟺ any function in $L^2(a,b)$
can be written as the F.S. associated to $\{X_1(x), \ldots, X_n(x)\}$.

<u>Corollary to theorem 3</u> $\boxed{\text{associated to the orthogonal set of } \cancel{\text{eigen}} \text{ eigenfunctions with symmetric B.C.}}$
For any f in $L^2(a,b)$ the Parseval equality holds.

<u>Proof of the Convergence theorems</u>: $\boxed{\text{Prove theorem 4, 4' first}}$

<u>Proof of theorem 2</u>: Uniform convergence with $\sin nx$ $\cos nx$

Consider the classical F.S. on functions of period 2π.
Assume f, f' are continuous.

$$A_n = \int_{-\pi}^{\pi} f(x)\cos nx \frac{dx}{\pi} \quad n=0,1,\ldots \qquad B_n = \int_{-\pi}^{\pi} f(x)\sin nx \frac{dx}{\pi} \quad n=1,\ldots$$

$$A'_n = \int_{-\pi}^{\pi} f'(x)\cos nx \frac{dx}{\pi} \quad n=0,1,\ldots \qquad B'_n = \int_{-\pi}^{\pi} f'(x)\sin nx \frac{dx}{\pi} \quad n=1,\ldots$$

Clearly by integration by parts

$$(*) \quad \begin{cases} A_n = -\dfrac{1}{n} B'_n & n \neq 0 \\[2ex] B_n = \dfrac{1}{n} A'_n \end{cases}$$

From Bessel's inequality

$$\sum_{n=1}^{\infty} |A_n'|^2 + |B_n'|^2 \le \int_a^b |f(x)|^2 dx < \infty$$

then

$$\sum_{n=1}^{\infty} \left(|A_n \cos nx| + |B_n \sin nx| \right) \le$$

$$\le \sum_{n=1}^{\infty} \left(|A_n| + |B_n| \right) = \sum_{n=1}^{\infty} \frac{1}{n} \left(|B_n'| + |A_n'| \right)$$

Recall ~~Cauchy~~ Schwarz's inequality

$$\sum_{n=1}^{\infty} a_n b_n \le \left(\sum a_n^2 \right)^{\frac{1}{2}} \left(\sum b_n^2 \right)^{\frac{1}{2}}$$

(Ex 5 p. 139)*

$$\le \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \left(\underbrace{\sum_{n=1}^{\infty} |B_n'|^2 + |A_n'|^2}_{\wedge| \; C} \right)^{\frac{1}{2}} \le C$$

So the series is absolutely convergent
now by Theorem 4 we already know that
the Fourier Series converges to f pointwise

$$f(x) = \sum_{n=0}^{\infty} A_n \cos nx + B_n \sin nx$$

* Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York: Wiley, 3 March 1992. ISBN: 0471548685.