Lecture # 14

Wave equation in an interval - Separation of variables

Consider ~~again~~ the wave equation with the Dirichlet data

$$(\ast)\quad \begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,t) = u(l,t) = 0 & \text{Dirichlet Data} \\ u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) & \text{Initial data} \end{cases}$$

We want to write the solution of $(\ast)$ in a way that works for any (x,t) and not having to go through the procedure of checking the periods and the odd-even extensions like we did in previous lectures.

The idea is to use "building blocks", that will be then summed.

Method of separation of variables

$$u(x,t) = X(x)\,T(t)$$

In order for u to be solution we need

$$T_{tt}(t)\,X(x) = c^2 X_{xx}(x)\,T(t)$$

$$\frac{T_{tt}(t)}{c^2\,T(t)} = \frac{X_{xx}(x)}{X(x)} = -\lambda$$

Because LHS depends only on t and RHS depends only on x it must be $\boxed{\lambda = \text{constant}}$

$\boxed{\underline{\text{Fact}}: \lambda > 0}$ if $X \neq 0$

Assume this fact for the moment and take

$$\lambda = \beta^2 \quad , \quad \beta > 0$$

then
$$\begin{cases} X'' + \beta^2 X = 0 \\ T'' + c^2\beta^2 T = 0 \end{cases}$$

this is a system of two uncoupled ODE which solutions are

$$T(t) = A\cos\beta c t + B\sin\beta c t$$

$$X(x) = C\cos\beta x + D\sin\beta x$$

A, B, C, D constant to be determined by data

$u(0,t) = u(l,t) = 0 \implies X(0) = X(l) = 0$

$X(0) \implies C = 0 \qquad X(x) = D\sin\beta x$

$X(l) = 0 \implies \qquad 0 = D\sin\beta l$

Clearly we don't want also $D=0$ so we ask

$$B \sin \beta l = 0$$

$$\Rightarrow \quad \beta l = \pi n \quad \Rightarrow \quad \beta = \frac{\pi n}{l}$$

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2$$

So there are many $X(x)$ that work

$$X_n(x) = D_n \sin \frac{n\pi x}{l} \qquad n = 1, 2, \ldots$$

So there are ∞ many solution for the Dirichlet problem

$$(D) \begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,t) = u(l,t) = 0 \end{cases}$$

Namely

$$(*) \quad u_n(x,t) = \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

where $A_n = AD_n$, $B_n = BD_n$ arbitrary constant.

In particular, due to linearity any finite sum

$$u_n(x,t) = \sum_{n=1}^{k} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

is a solution for the Dirichlet problem (D).

Now if we want to solve (DI) then we need

$$\boxed{u_k(x,0) = \sum_{n=1}^{k} A_n \sin \frac{n\pi x}{\ell} = \phi(x)}$$

$$u_{kt}(x,t) = \sum_{n=1}^{k} \left(-A_n \sin \frac{n\pi ct}{\ell} \cdot \frac{n\pi c}{\ell} + B_n \cos \frac{n\pi ct}{\ell} \cdot \frac{n\pi c}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

$$\boxed{u_{kt}(x,0) = \sum_{n=1}^{k} B_n \frac{n\pi c}{\ell} \sin \frac{n\pi x}{\ell} = \psi(x)}$$

<u>Remark</u>: If $\phi(x)$ and $\psi(x)$ are of the special form above the solution is found.

Let's write the reason in an inverse way:

Assume $\phi(x) = \sum_{n=1}^{k} A_n \sin \frac{n\pi}{\ell} x$

Notice that ϕ is periodic of period 2ℓ and

$$\psi(x) = \sum_{n=1}^{k} D_n \sin \frac{n\pi x}{\ell}$$

then if we set $B_n = D_n \cdot \frac{\ell}{n\pi c}$

the solution for (DI) is given by (*).

Definition:

The number $\frac{n\pi c}{l}$ = frequencies of the solutions.

Diffusion equation

$$(DI)\begin{cases} u_t = k u_{xx} & 0<x<l \quad 0<t<\infty \\ u(0,t) = u(l,t) = 0 \\ \cancel{u(x,0)} = \\ u(x,0) = \phi(x) \end{cases}$$

$$u(x,t) = X(x)\,T(t)$$

$$X(x)\,T_t(t) = k\, X_{xx}(x)\,T(t)$$

$$\frac{T_t(t)}{k\,T(t)} = \frac{X_{xx}(x)}{X(x)} = -\lambda$$

$$\begin{cases} X'' = -\lambda X & 0<x<l \\ X(0) = X(l) = 0 \end{cases}$$

$$X_n(x) = B_n \sin\frac{n\pi x}{l}$$

$$T'(t) = -\left(\frac{n\pi}{l}\right)^2 k\, T(t)$$

$$T(t) = A\, e^{-\left(\frac{n\pi}{l}\right)^2 k t}$$

$$(D)\quad u_k(x,t) = \sum_{n=1}^{k} A_n\, e^{-\left(\frac{n\pi}{l}\right)^2 k t} \sin\frac{n\pi x}{l}$$

and this solves (DI) if

$$\phi(x) = \sum_{n=1}^{N} A_n \sin \frac{n\pi x}{\ell}$$

<u>Remarks</u>: 1) Notice that thanks to the fast decay of the fact $e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$ one can actually take the infinite sum $\sum_{n=1}^{\infty}$!

2) In the next few lectures we we show the following: any continuous function $\phi(x)$ periodic $[0, \ell]$ can be expand as an infinite sum

$$\phi(x) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi x}{\ell}$$

for appropriate ~~...~~ constant values A_n.

Then the solution we found above is really the general case, not the special one.

<u>Fact 1</u>: Prove that all possible λ s.t.

$$\begin{cases} X''(x) = -\lambda X & X \neq 0 \\ X(0) = X(\ell) = 0 \end{cases}$$

~~...~~ ~~...~~ are positive.

Proof:

Case 1: $\lambda = 0$, then

$X''(x) = 0 \implies X(x) = Ax + B$

$X(0) = B = 0$

$X(l) = Al + B = 0 \implies A, B \equiv 0$

$\implies X \equiv 0$ not allowed.

Case 2: $\lambda < 0 \implies \lambda = -\gamma^2$ $\quad \cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$X'' = \gamma^2 X$ $\quad \sinh(x) = \dfrac{e^x - e^{-x}}{2}$

then $X(x) = C\cosh\gamma x + D\sinh\gamma x$

$X(0) = C = 0 \implies X(x) = D\sinh\gamma x$

$X(l) = D\sinh\gamma l = 0 \implies D = 0$

Since $\gamma l \neq 0$

So again $X(x) \equiv 0$ not allowed.

Case 3: λ complex

let γ and $-\gamma$ s.t. $(\gamma^2) = (-\gamma)^2 = \lambda$

$X(x) = Ce^{\gamma x} + De^{-\gamma x}$

$X(0) = C + D = 0$

$X(l) = Ce^{\gamma l} + De^{-\gamma l} = 0$

$$-De^{\gamma l}+De^{-\gamma l}=0$$

$$D\left(e^{-\gamma l}-e^{\gamma l}\right)=0$$

$$De^{\gamma l}\left(e^{-2\gamma l}-1\right)=0 \Leftarrow$$

$$e^{2\gamma l}=1$$

Now if $\gamma=\alpha+i\beta$

$$e^{2\alpha l}e^{i2l\beta}=1$$

$\Rightarrow$ $\alpha=0$ and $2l\beta=2\pi n$

$$\beta=\frac{\pi n}{l} \Rightarrow$$

$$\gamma=\frac{i\pi n}{l} = -\lambda=\gamma^2=-\left(\frac{\pi n}{l}\right)^2$$

so $\lambda=\left(\frac{\pi n}{l}\right)^2$ not complex. $\Rightarrow$ II

so $\boxed{\lambda>0}$

<u>the Neumann Condition</u>

$$(N\,I)\begin{cases} u_{tt}-c^2u_{xx}=0 \\ u_x(0,t)=u_x(l,t)=0 \\ u(x,0)=\phi \\ u_t(x,0)=\psi \end{cases}$$

Consider now

$$(N)\quad\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u_x(0,t) = u_x(\ell,t) = 0 \end{cases}$$

we look for

$$u(x,t) = X(x)\,T(t)$$

$$\begin{cases} -X'' = \lambda X & X \not\equiv 0 \\ X'(0) = X'(\ell) = 0 \end{cases}$$

Now in this case it is not guaranteed that $\lambda > 0$

<u>Case 1</u>: $\lambda = \beta^2 > 0$

$$X(x) = C\cos\beta x + D\sin\beta x$$

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x$$

$$X'(0) = D\beta = 0 \implies D = 0$$

$$X'(\ell) = -C\beta\sin\beta\ell = 0$$

Since $C = 0$ is not allowed and $\beta > 0$

$$\beta\ell = \pi n$$

$$\beta = \frac{\pi n}{\ell} \implies$$

$$\begin{cases} \lambda = \left(\dfrac{\pi n}{l}\right)^2 & \text{eigenvectors} \\ X_n(x) = \cos\dfrac{n\pi x}{l} & \text{eigenfunctions} \quad n = 1, 2 \ldots \end{cases}$$

Case 2 $\lambda = 0$

$$X''(x) = 0 \qquad X(x) = Ax + B$$

$$X'(x) = A \qquad X'(0) = X'(l) = 0 = A$$

$$\boxed{X(x) = B}$$

Case 3 $\lambda < 0$ or λ = complex the only

function $X(x)$ that works is $X = 0$ not allowed

(Prove this at home like in the Dirichlet case)

Conclusion:

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2 \qquad n = 0, 1, 2 \ldots$$

this takes care of $\lambda = 0$

~~[illegible]~~

Let's go back to

$$T''(t) = \lambda c^2 T(t)$$

if $\lambda = 0$ $\quad T''(t) = 0 \quad T(t) = A + Bt$

So for $\lambda = 0$ the solution to (N I) becomes (11)

1) $u(x,t) = \{A + Bt\}$ because in this case $X(x) =$ const.

~~and~~ $\cancel{u(x,0) = A}$ $\cancel{u_t(x,t) = B}$ for $\lambda > 0$

~~So this solves the equation~~

2) $$u_k(x,t) = \sum_{n=1}^{k} \left(A_n \cos\frac{n\pi c t}{\ell} + B_n \sin\frac{n\pi c t}{\ell} \right) \cos\frac{n\pi x}{\ell}$$

We can combine both to get

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi c t}{\ell} + B_n \sin\frac{n\pi c t}{\ell} \right) \cdot \cos\frac{n\pi x}{\ell}$$

~~To see~~

this comes from
some normalization
that will be addressed later

To solve (N I) we then need

$$u_t(x,t) = \frac{1}{2} B_0 + \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi c}{\ell} \sin\frac{n\pi c t}{\ell} + B_n \frac{n\pi c}{\ell} \cos\frac{n\pi c t}{\ell} \right) \cos\frac{n\pi x}{\ell}$$

$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{\ell}$$

$$\psi(x) = \frac{1}{2} B_0 + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{\ell} \cos \frac{n\pi x}{\ell}$$