Now consider the solution $Q(x,t)$ of the IVP

Lecture # 8

$$\begin{cases} u_t = k u_{xx} \\ u|_{t=0} = Q(0,x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \end{cases}$$

If we dilate $Q(x) \rightarrow Q(\lambda x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} = Q(x)$

So we also expect it's solution not to change w.r.t. dilation, i.e. so that

$$Q(x,t) = Q(\lambda x, \lambda^2 t)$$

So we look for $Q(x,t) = g\left(\underbrace{\frac{x}{\sqrt{4kt}}}_{\text{normalization factor}}\right)$

Now by chain rule $\frac{x}{\sqrt{4kt}} = p \quad t > 0$

$$Q_t = \frac{dg}{dp} \cdot \frac{dp}{dt} = g'(p)\left(-\frac{1}{2t}\frac{x}{\sqrt{4kt}}\right)$$

$$Q_x = \frac{dg}{\partial p}\frac{dp}{\partial x} = \frac{g'(p)}{\sqrt{4kt}}$$

$$Q_{xx} = \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt} g''(p)$$

$$0 = Q_t - Q_{xx} = \frac{1}{t}\left[-\frac{1}{2} p g'(p) - \frac{1}{4} g''(p)\right]$$

then $\boxed{g''(p) + 2p g'(p) = 0}$

Now this is an ODE which solution is

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$$g(\xi) = c_1 \int_0^{\xi} e^{-\beta^2} d\beta + c_2$$

so $$Q(x,t) = c_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp + c_2 \quad \forall t > 0$$

We need to check $\lim t \to 0$

if $x > 0$ $$\lim_{t \to 0^+} c_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp + c_2$$

$$= c_1 \int_0^{\infty} e^{-p^2} dp + c_2$$

if $x < 0$ $$\lim_{t \to 0^+} [\quad] = -c_1 \int_{-\infty}^{0} e^{-p^2} dp + c_2$$

but $\int_0^{\infty} e^{-p^2} dp = \frac{\sqrt{\pi}}{2}$ and $\int_{-\infty}^{0} e^{-p^2} dp = -\frac{\sqrt{\pi}}{2}$

$$\lim_{t \to 0^+} [\quad] = \begin{cases} c_1 \frac{\sqrt{\pi}}{2} + c_2 & x > 0 \\ -c_1 \frac{\sqrt{\pi}}{2} + c_2 & x < 0 \end{cases} = Q(x,0) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\begin{cases} c_1 \frac{\sqrt{\pi}}{2} + c_2 = 1 \\ -c_1 \frac{\sqrt{\pi}}{2} + c_2 = 0 \end{cases} \Longrightarrow \quad c_2 = \frac{1}{2} \quad c_1 = \frac{1}{\sqrt{\pi}}$$

$$\boxed{Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp}$$

Now define $S = \frac{\partial Q}{\partial x}$ Source function property b)

Remark: a) S solves the diffusion equation
b) $\lim_{t \to 0^+} \frac{\partial Q}{\partial x}(t) = \delta_0$

We will see this later

First we observe that

$$\partial_x Q(x,t) = \partial_x\left(\frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4tk}}} e^{-p^2}\,dp\right)$$

$$= \boxed{\frac{1}{\sqrt{\pi}}\, e^{-\left(\frac{x}{4tk}\right)^2}\cdot\frac{1}{\sqrt{4tk}} = S(x,t) \text{ source function!}}$$

~~Take a test function $\phi(x)$~~

$$\cancel{\lim_{t\to 0}\left(\frac{1}{\sqrt{\pi}}\, e^{-\frac{x^2}{4t}}\,\frac{1}{\sqrt{4t}}, \varphi\right)} =$$

$$= \cancel{\lim_{t\to 0}\int} \quad \text{so if } x \neq 0$$

$$\lim_{t\to 0^+}\frac{1}{\sqrt{\pi}}\, e^{-\frac{x^2}{4t}}\,\frac{1}{\sqrt{4t}} = 0$$

because exponential decays more than any polinomial

$x = 0$

$$\partial_x Q(0,t) = \frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{4t}} \qquad t \neq 0$$

so it looks syngular, but if one uses the theory of distribution...

Now assume that $\phi(x)$ is a given function $\underbrace{}_{\text{st. } \lim_{|x|\to\infty}\phi(x) = 0}$ the define

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy \qquad t > 0.$$

By (d) u is also a solution

<u>Claim:</u> u is the unique solution of

$$\begin{cases} u_t = k u_{xx} \\ u|_{t=0}(x) = \phi(x) \end{cases} \qquad \lim_{|x| \to \infty} \phi(x) = 0$$

Proof: We only need to check the value of u at $t=0$

$$u(x,t) = \int_{-\infty}^{\infty} \partial_1 Q(x-y,t)\, \phi(y)\, dy$$

$$= -\int_{-\infty}^{\infty} \partial_y Q(x-y,t)\, \phi(y)\, dy$$

$$= \underbrace{Q(x-y,t)\,\phi(y)\Big|_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} Q(x-y,t)\, \phi'(y)\, dy$$

So

$$u(x,0) = \int_{-\infty}^{\infty} \underbrace{Q(x-y,0)}_{= \begin{cases} 1 \text{ if } x-y>0 \\ 0 \text{ if } x-y<0 \end{cases}} \phi'(y)\, dy$$

$$= \int_{\{x-y>0\}} \phi'(y)\, dy = \int_{-\infty}^{x} \phi'(y)\, dy = \phi(x)$$

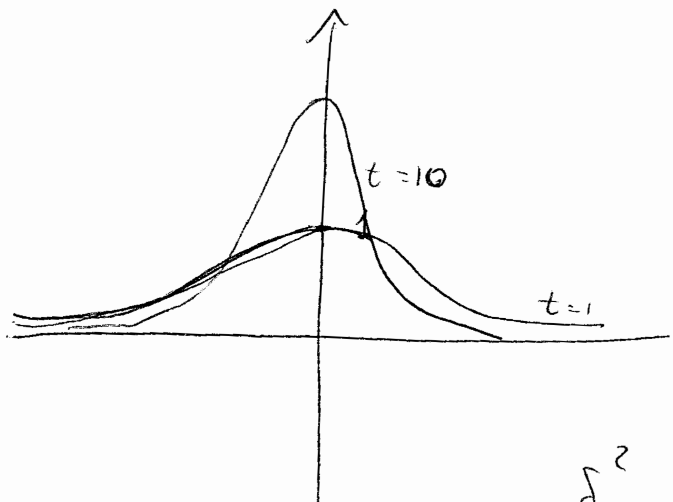
So

$$\boxed{u(x,y) = \frac{1}{\sqrt{4\pi t k}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}}\, \phi(y)\, dy}$$

Q.E.D.

The function $S(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt}$ $\quad t>0$

is called source function, Green's function, fundamental solution or propagator



$$f(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{t}}$$

$$f'(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{t}} \left(-\frac{2x}{t}\right)$$

$t=1 \qquad f(x,1) = e^{-x^2}$

$t=10$

$$\max_{|x|>\delta} S(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{\delta^2}{t}} \xrightarrow[t\to 0^+]{} 0, \quad \text{Also} \int S(x,t)\,dx = \frac{1}{2\sqrt{\pi k t}} \int e^{-\frac{x^2}{4kt}} dx = 1$$

this also support the fact that viewed as a distribution $S(x,t) \to \delta_0$.

Problem: Solve $\begin{cases} u_t = k u_{xx} & k>0 \\ u(x,0) = x^2 \end{cases}$

$$u(x,t) = \int \frac{1}{2\sqrt{\pi k t}} e^{-\frac{(x-y)^2}{4kt}} y^2\,dy$$

$$\frac{x-y}{\sqrt{4kt}} = p \qquad dp = -\frac{1}{\sqrt{4kt}}\,dy$$

$$= \frac{1}{\sqrt{\pi}} \int dp\, e^{-p^2} (x - p\sqrt{4kt})^2\, dp$$

$$= x^2 \int \frac{1}{\sqrt{\pi}} e^{-p^2} dp + \frac{2x\sqrt{4kt}}{\sqrt{\pi}} \int e^{-p^2} p \, dp \qquad (*)$$

$$+ \frac{4kt}{\sqrt{\pi}} \int e^{-p^2} p^2 dp$$

$p^2 = \zeta \quad d\zeta = 2p\,dp$ → this is not a trivial integral!

other way: u_{xxx} solves $u_t = k u_{xx}$ if u does (linearity)

$$u_{xxx}\big|_{t=0} = (x^2)_{xxx} = 0$$

so by uniqueness $u_{xxx}(x,t) = 0 \Rightarrow u_{xx}(x,t) = 2A(t)$

$u_x(x,t) = 2A(t)x + B(t)$, $u(x,t) = A(t)x^2 + B(t)x + C(t)$

$$\boxed{A'(t)x^2 + B'(t)x + C'(t) = k\,2A(t)}$$ equation $u_t = k u_{xx}$

$$\begin{aligned} A'(t) &\equiv 0 & & & A(t) &= c_0 \\ B'(t) &\equiv 0 & &\Rightarrow & B(t) &= c_1 \\ C'(t) &= k\,2A(t) & & & 2kc_0t + c_2 &= C(t) \end{aligned}$$

on the other hand

$$u(x,0) = A(0)x^2 + B(0)x + C(0) \equiv x^2$$

$$\begin{aligned} B(0) &= 0 &&\Rightarrow c_1 = 0 = B(t) \\ C(0) &= 0 &&\Rightarrow c_2 = 0 \\ A(0) &= 1 &&\Rightarrow c_0 = 1 \end{aligned}$$

$C(t) = 2kt \quad A(t) = 1$

$$u(x,t) = x^2 + 2kt \qquad (**)$$

If we compare $(*)$ with $(**)$ we have

$$\boxed{\frac{2}{\sqrt{\pi}} \int e^{-\rho^2} \rho^2 \, d\rho = 1}$$

Comparison of wave and diffusion equation

Fundamental properties

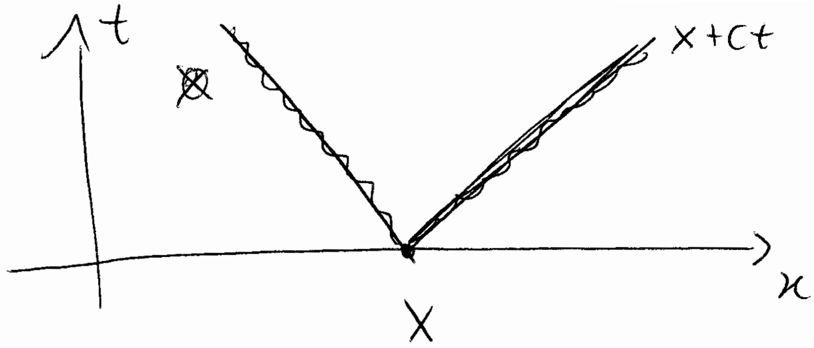
Property:	Wave	Diffusion
i) Speed of propagation	Finite ($\leq c$)	Infinite
ii) Singularities for $t>0$ (see initial data with no velocity)	Transported along characteristics (speed $= c$)	Lost immediately
iii) Well posedness for $t>0$?	Yes	Yes (nice data)
iv) Well posedness for $t<0$?	Yes	No
v) Maximum principle	No	Yes
vi) Behavior as $t \to \infty$	$E = \text{const}$	$E \searrow$

Some explanations

For wave:

if $\phi(x) = u(0,x)$, u solves $u_{tt} - c^2 u_{xx} = 0$

then we know that $u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$



for diffusion equation $u_t = k u_{xx}$ $\quad u(x,0) = \phi(x)$

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int e^{-\frac{(x-y)^2}{4kt}} \phi(y)\,dy$$

all derivatives for u continuous on — so ϕ maybe singular but u is not!

~~so if supp ϕ is in $[x_0-\delta, x_0+\delta]$ then $x-y$~~

~~$x-x_0-\delta < x-y < x-x_0+\delta$~~

so the value of u at (x,t) depends on the value of ϕ everywhere.

Similarly anything changed for ϕ at a certain point x_0 effects the values of u everywhere and at all times.

These facts take care of i) and ii)

iii) has been already discussed.

iv) If one changes variable $u(x,t) \longrightarrow t = -\tau$

then $\quad u(x,-\tau) = v(x,\tau)$

$v_{\tau\tau} = u_{tt}(x,-\tau)$ $\quad$ so also $\quad v_{\tau\tau} - c^2 v_{xx} = 0 \quad \tau > 0$

$v_{xx} = u_{xx}(x,-\tau)$

This can be solved hence $u(x,t)$, $t<0$ can be solved.

On the other hand

$$v_t = -u_{tt}(x,-t) \qquad \text{so} \qquad v_t - k v_{xx} = 0$$

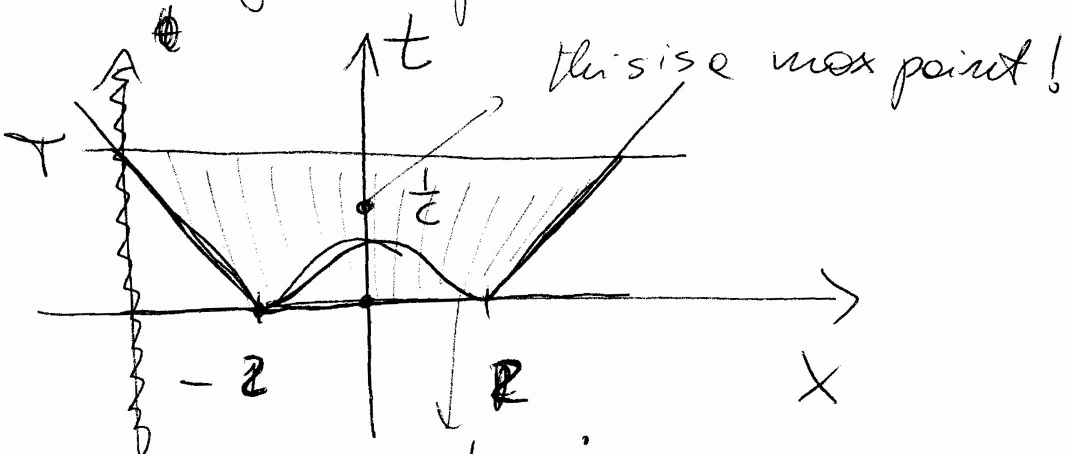
$$v_{xx} = u_{xx}(x,-t)$$

and one can see that the source function is now

$$\frac{1}{2\sqrt{\pi k t}}\, e^{\frac{x^2}{4kt}}$$ which is not integrable $t>0$

→ not integrable.

v) That this is not true for the wave comes from the following example



$\phi = u(0,x)$

$\phi = 0$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \underset{1}{\psi(s)}\, ds$$

$x_0 = 0$

$ct = 1 \qquad t = \frac{1}{c}$

Summarizing:

1) Classification of PDE
- linearity, homogeneity, inhomogeneity
- non linearity
- order

2) Solution to linear 1st order PDE in $\mathbb{R}^2$ (homogeneous + inhomogeneous)

4) ~~Classification~~ types of 2nd order linear PDE
- elliptic
- Hyperbolic
- parabolic

2 initial and boundary conditions

3) Well-posedness,

6) Wave equation in $\mathbb{R}^1$
- solution of initial value problem
- causality
- Energy
- Source function

7) Diffusion Equation
- Maximum principle
- uniqueness
- Energy
- Source function
- Solution of the IVP on $\mathbb{R}$.