Optimization-Based Animation

V. Milenkovic, H. Schmidt

SIGGRAPH 2001

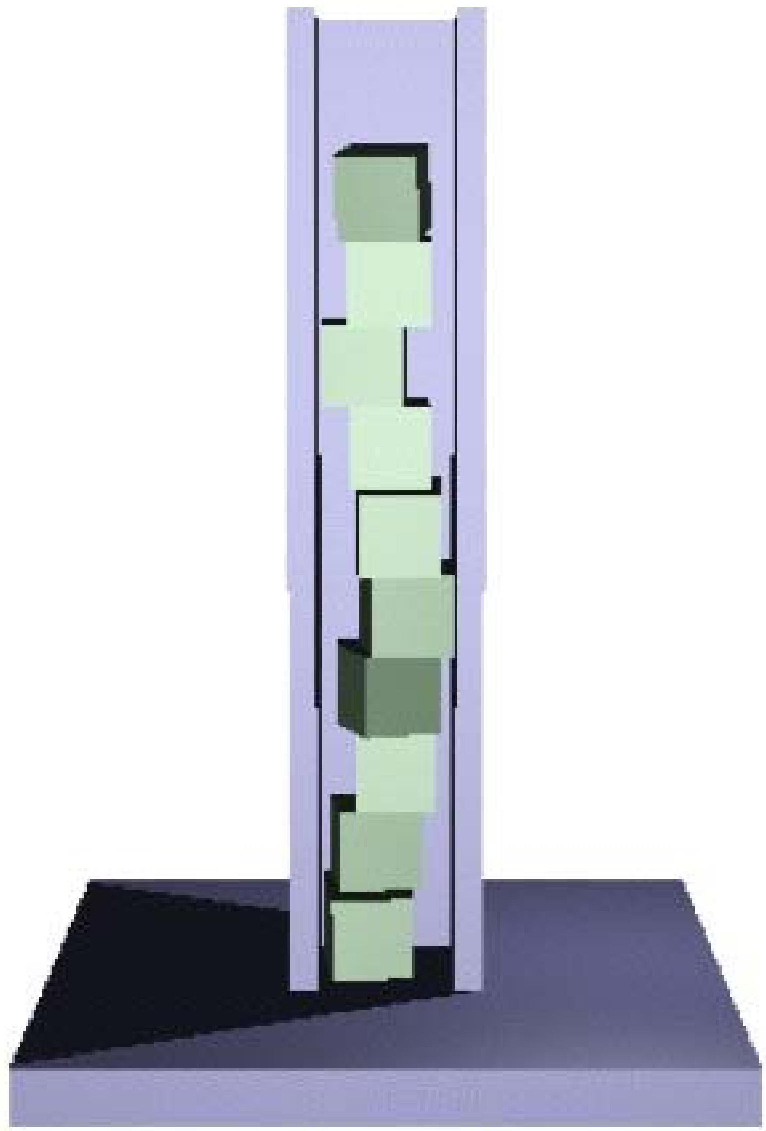

Figure 1: Simulating stacked bodies is known to be difficult.

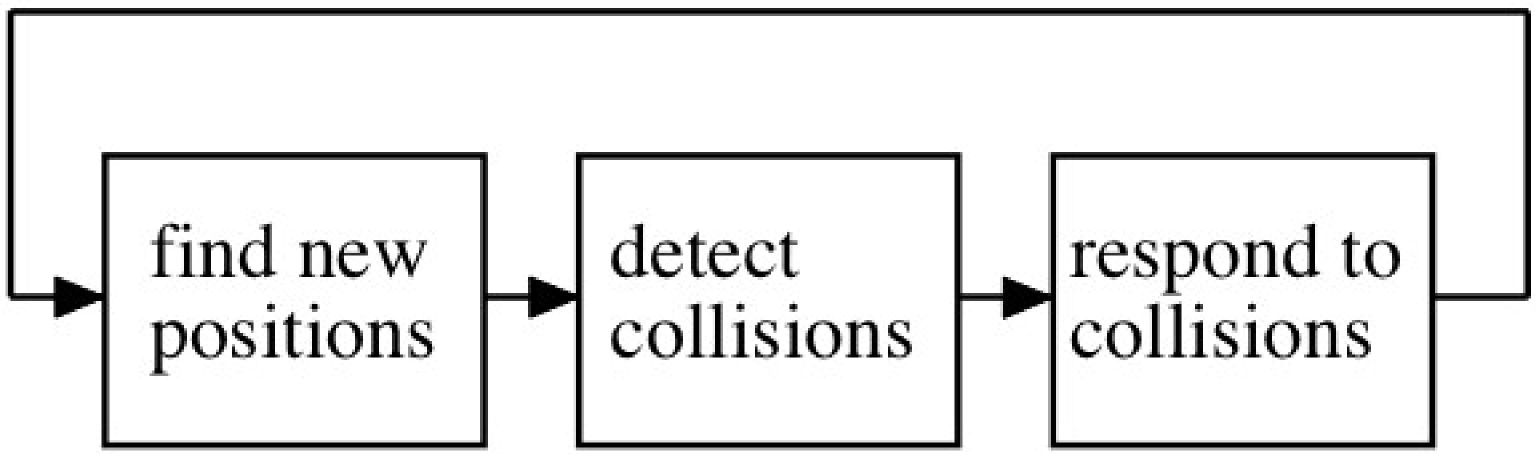

Figure 2: The simulation loop.

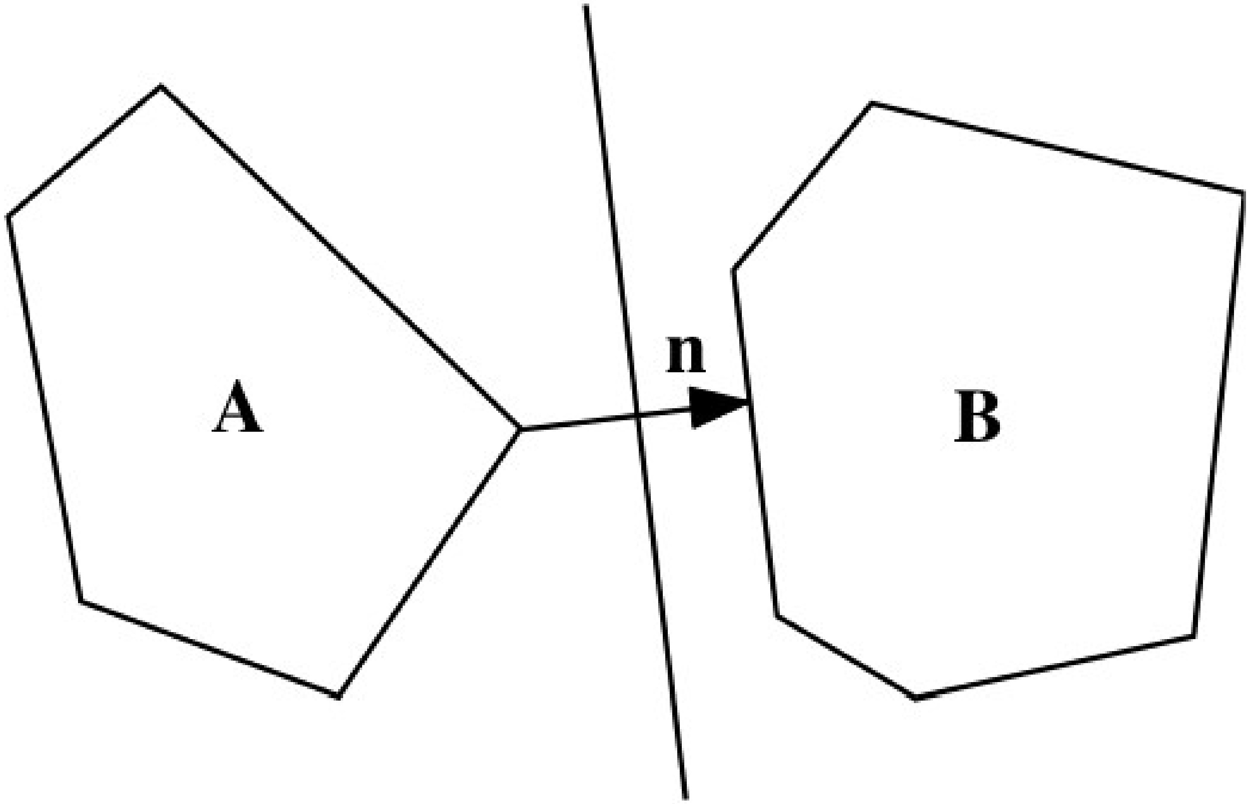

Figure 3: Separating plane between bodies **A** and **B** with normal vector **n**.

$$\forall \mathbf{q}_a \in \mathbf{A}, \mathbf{n} \cdot \mathbf{q}_a \leq d \quad \text{and} \quad \forall \mathbf{q}_b \in \mathbf{B}, \mathbf{n} \cdot \mathbf{q}_b \geq d. \qquad (1)$$

Objective

$$\Delta \mathbf{x} = \mathbf{x}^{\text{per}} - \mathbf{x} \quad \text{and} \quad \Delta \mathbf{R} = \mathbf{R}^{\text{per}} \mathbf{R}^{-1}. \tag{2}$$

$$\Delta \mathbf{x}^{\text{tgt}} = \mathbf{x}^{\text{tgt}} - \mathbf{x} \quad \text{and} \quad \Delta \mathbf{R}^{\text{tgt}} = \mathbf{R}^{\text{tgt}} \mathbf{R}^{-1}. \tag{3}$$

$$\Delta\Delta \mathbf{x} = \Delta \mathbf{x} - \Delta \mathbf{x}^{\text{tgt}} \quad \text{and} \quad \Delta\Delta \mathbf{r} = \Delta \mathbf{r} - \Delta \mathbf{r}^{\text{tgt}}. \tag{4}$$

$$\sum_{i=1}^{k} \Delta\Delta \mathbf{x}_i \cdot \Delta\Delta \mathbf{x}_i + D_i^2 \Delta\Delta \mathbf{r}_i \cdot \Delta\Delta \mathbf{r}_i, \tag{5}$$

$$\sum_{i=1}^{k} m_i \Delta\Delta \mathbf{x}_i \cdot \Delta\Delta \mathbf{x}_i + \Delta\Delta \mathbf{r}_i^T \mathbf{I}_i \Delta\Delta \mathbf{r}_i. \tag{6}$$

Linearizing the Separation Constraints

$$\Delta \mathbf{n} = \mathbf{n}^{\text{per}} - \mathbf{n} \approx \delta_x \mathbf{n}_x + \delta_y \mathbf{n}_y, \tag{7}$$

$$\mathbf{q}_a = \mathbf{R}_a \mathbf{q}_a^{\text{body}} + \mathbf{x}_a \quad \text{and} \quad \mathbf{q}_a^{\text{per}} = \mathbf{R}_a^{\text{per}} \mathbf{q}_a^{\text{body}} + \mathbf{x}_a^{\text{per}}.$$

$$\Delta \mathbf{R} \mathbf{w} \approx \mathbf{w} + \Delta \mathbf{r} \times \mathbf{w}:$$

$$\mathbf{q}_a^{\text{per}} = \Delta \mathbf{R}_a (\mathbf{q}_a - \mathbf{x}_a) + \mathbf{x}_a^{\text{per}}. \tag{8}$$

$$\mathbf{q}_a^{\text{per}} \approx \mathbf{q}_a + \Delta \mathbf{x}_a + \Delta \mathbf{r}_a \times (\mathbf{q}_a - \mathbf{x}_a). \tag{9}$$

Linearizing the Separation Constraints

$$(\mathbf{n} + \delta_x \mathbf{n}_x + \delta_y \mathbf{n}_y) \cdot (\mathbf{q}_a + \Delta\mathbf{x}_a + \Delta\mathbf{r}_a \times (\mathbf{q}_a - \mathbf{x}_a)) \leq d. \quad (10)$$

$$\begin{array}{ll} \mathbf{n} \cdot \Delta\mathbf{x}_a - (\mathbf{n} \times (\mathbf{q}_a - \mathbf{x}_a)) \cdot \Delta\mathbf{r}_a & \\ +(\mathbf{n}_x \cdot \mathbf{q}_a)\delta_x + (\mathbf{n}_y \cdot \mathbf{q}_a)\delta_y - d & \leq \quad -\mathbf{n} \cdot \mathbf{q}_a. \end{array} \quad (11)$$

$$\begin{array}{ll} \mathbf{n} \cdot \Delta\mathbf{x}_b - (\mathbf{n} \times (\mathbf{q}_b - \mathbf{x}_b)) \cdot \Delta\mathbf{r}_b & \\ +(\mathbf{n}_x \cdot \mathbf{q}_b)\delta_x + (\mathbf{n}_y \cdot \mathbf{q}_b)\delta_y - d & \geq \quad -\mathbf{n} \cdot \mathbf{q}_b. \end{array} \quad (12)$$

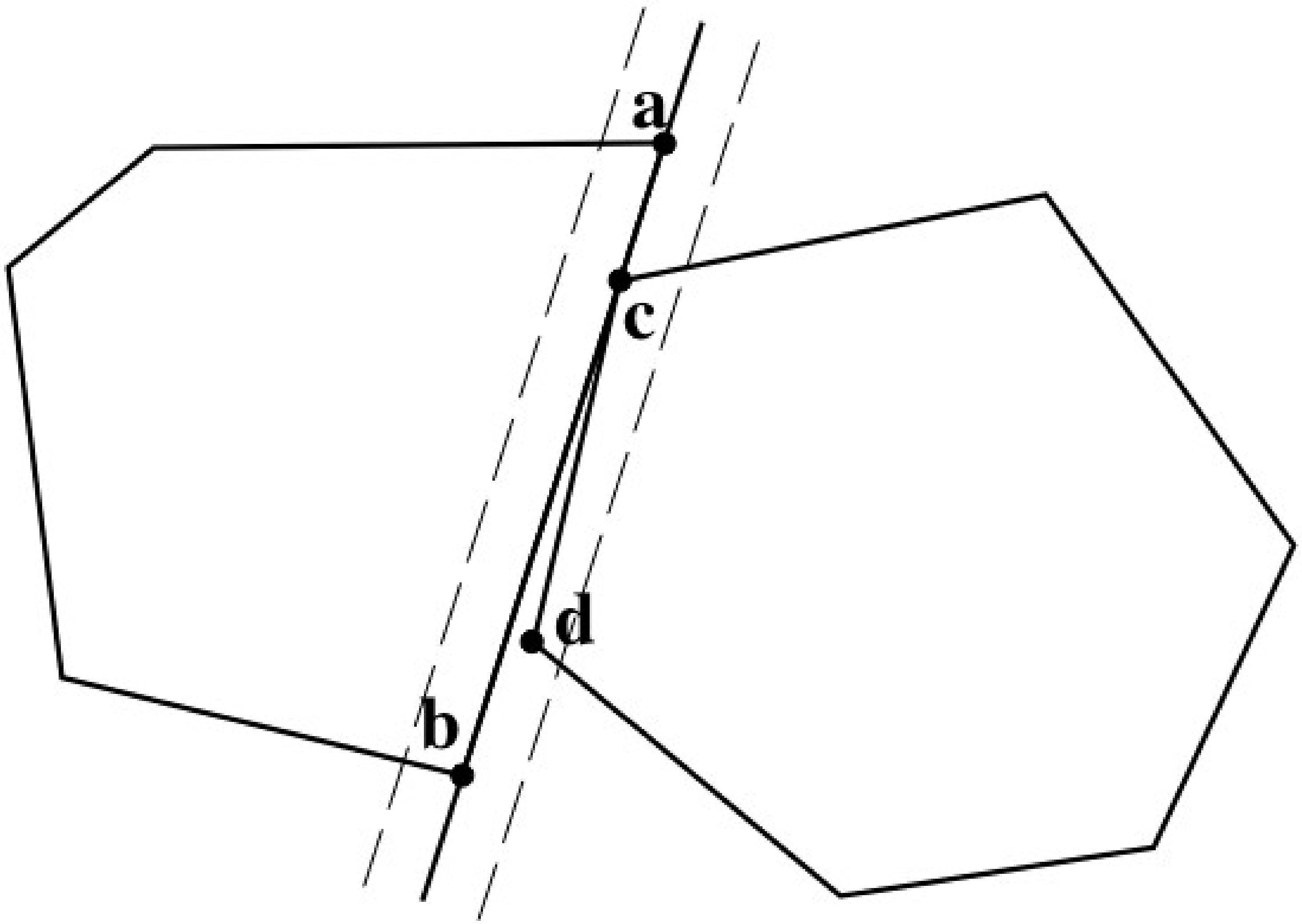

Figure 4: Explanation of critical vertices.

Input: current and target positions for each body
Output: new current position for each body

$S \leftarrow \emptyset$
repeat
  find current close pairs and add to S
  for all close pairs $P = \langle \mathbf{A}, \mathbf{B}, C \rangle \in S$
    calculate the separating plane $\langle \mathbf{n}, d \rangle$ for $\mathbf{A}$ and $\mathbf{B}$
    find current critical vertices of $\mathbf{A}$ and $\mathbf{B}$ with respect to $\langle \mathbf{n}, d \rangle$ and add them to C
  solve QP
  if infeasible
    rollback all body positions
  else
    update the current positions
while QP was infeasible **or**
  critical vertices were added **or**
  objective was improved

Table 1: Position update algorithm.

Bounds

$$d = \frac{1}{2}\left(\max_{\mathbf{q}_a \in A} \mathbf{n} \cdot \mathbf{q}_a + \min_{\mathbf{q}_b \in B} \mathbf{n} \cdot \mathbf{q}_b\right), \qquad (13)$$

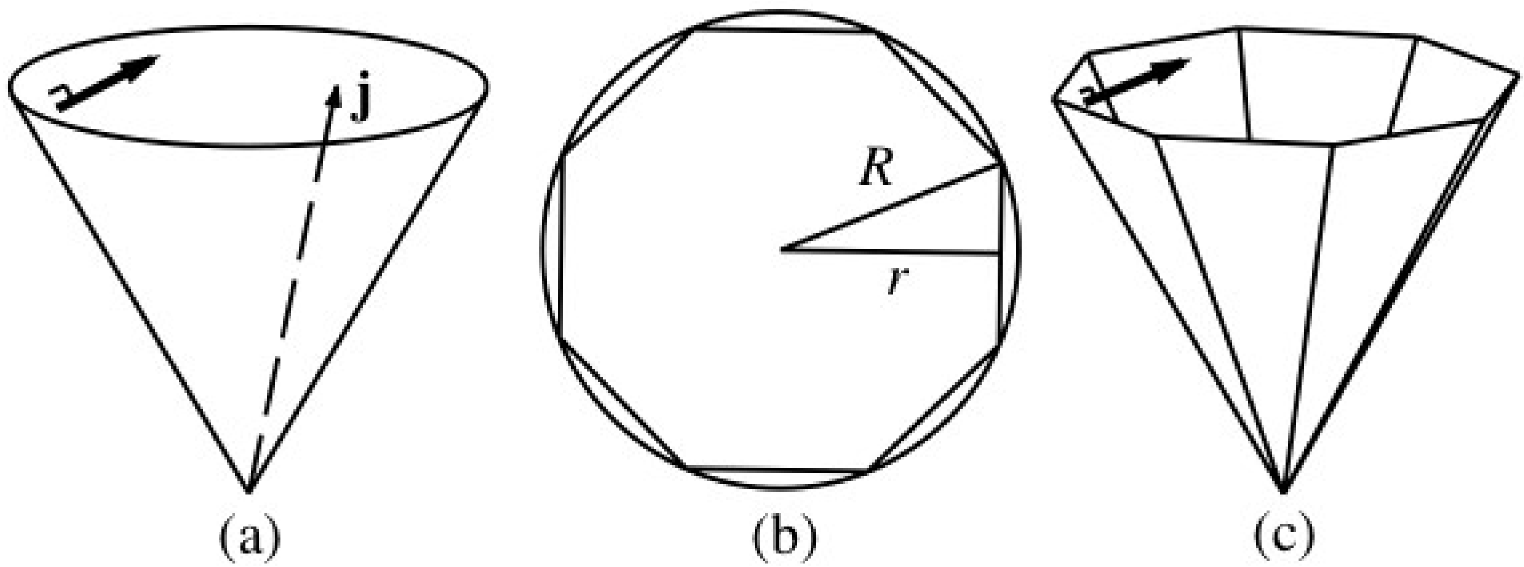

Figure 5: Friction cone and regular octagonal approximation. Ratio r/R of inner to outer radius is $\cos \pi/8$.

Momentum Update

$$(\mathbf{n}_x \cdot \mathbf{j})^2 + (\mathbf{n}_y \cdot \mathbf{j})^2 \leq \mu^2 (\mathbf{n} \cdot \mathbf{j})^2 \quad \text{and} \quad \mathbf{n} \cdot \mathbf{j} \geq 0. \tag{14}$$

$$\mathbf{u}_h = \mu \cos \frac{\pi}{8} \mathbf{n} + \cos \frac{\pi h}{4} \mathbf{n}_x + \sin \frac{\pi h}{4} \mathbf{n}_y, \tag{15}$$

$$\mathbf{u}_h \cdot \mathbf{j} \geq 0, \qquad \text{for} \qquad h = 0, 1, 2, \ldots, 7. \tag{16}$$

$$v(\mathbf{q}) = \mathbf{n} \cdot ((\mathbf{v}_b + \omega_b \times (\mathbf{q} - \mathbf{x}_b)) - (\mathbf{v}_a + \omega_a \times (\mathbf{q} - \mathbf{x}_a))). \tag{17}$$

$$m_b^{-1} \mathbf{j} \qquad \text{and} \qquad \mathbf{I}_b^{-1} ((\mathbf{q} - \mathbf{x}_b) \times \mathbf{j}) \tag{18}$$

$$m_a^{-1} \mathbf{j} \qquad \text{and} \qquad \mathbf{I}_a^{-1} ((\mathbf{q} - \mathbf{x}_a) \times \mathbf{j}) \tag{19}$$

$$\text{if } v^- \geq 0 \text{ then } v^+ \geq 0 \text{ else } v^+ \geq -\epsilon \cdot v^-. \tag{20}$$

Force or Acceleration Calculation

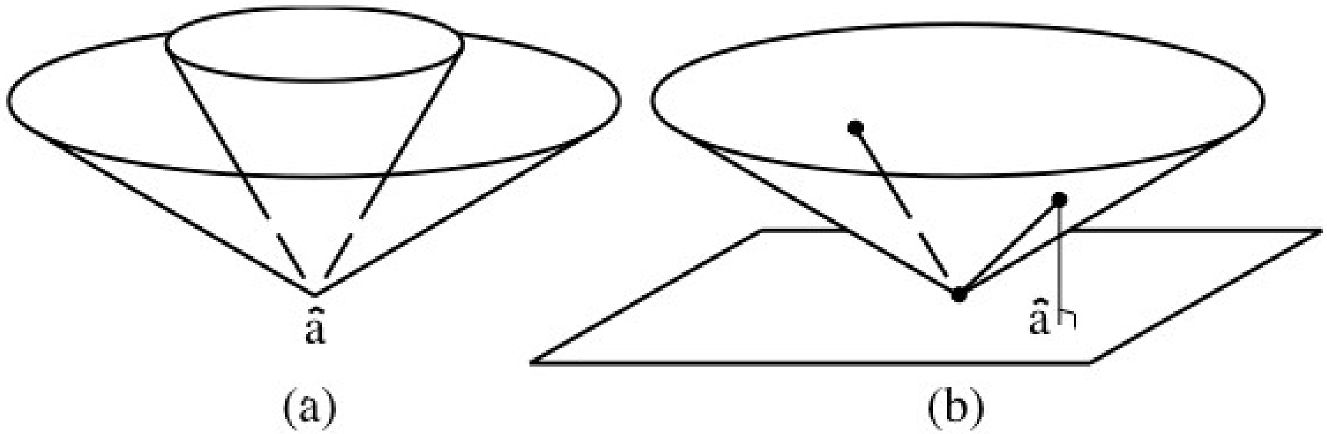

Figure 6: Acceleration cone vs. friction cone. If $\mathbf{a}$ lies on side of cone, new $\hat{\mathbf{a}}$ is projection to $\mathbf{n}_x, \mathbf{n}_y$ plane.

$$\mathbf{a}(\mathbf{q}) = (\mathbf{a}_b + \alpha_b \times (\mathbf{q} - \mathbf{x}_b)) - (\mathbf{a}_a + \alpha_a \times (\mathbf{q} - \mathbf{x}_a)). \quad (21)$$

$$\mathbf{n} \cdot \mathbf{a}(\mathbf{q}) \geq 0. \quad (22)$$

Force or Acceleration Calculation

$$\sum_{i=1}^{k} -m_i \mathbf{g} \cdot \mathbf{a} + \frac{1}{2} m_i \mathbf{a} \cdot \mathbf{a} + \frac{1}{2} \alpha_i^T \mathbf{I}_i \alpha_i. \quad (23)$$

$$\hat{\mathbf{a}}(\mathbf{q}) = \mathbf{a}(\mathbf{q}) - (\mathbf{n} \cdot \mathbf{a}(\mathbf{q}))\mathbf{n}. \quad (24)$$

$$\left(\mu^{-1} \cos \frac{\pi}{8} \mathbf{n} + \cos \frac{\pi h}{4} \mathbf{n}_x + \sin \frac{\pi h}{4} \mathbf{n}_y\right) \cdot (\mathbf{a}(\mathbf{q}) - \hat{\mathbf{a}}(\mathbf{q})) \geq 0. \quad (25)$$

for each frame
    update positions and find contact points
    apply impulses
    calculate new body accelerations

Joints

Figure 7: Simple multi-body pendulum made of six sticks.

$$\mathbf{q}_a + \Delta\mathbf{x}_a + \Delta\mathbf{r}_a \times (\mathbf{q}_a - \mathbf{x}_a) = \mathbf{q}_b + \Delta\mathbf{x}_b + \Delta\mathbf{r}_b \times (\mathbf{q}_b - \mathbf{x}_b), \quad (26)$$

Non-Convex Solids

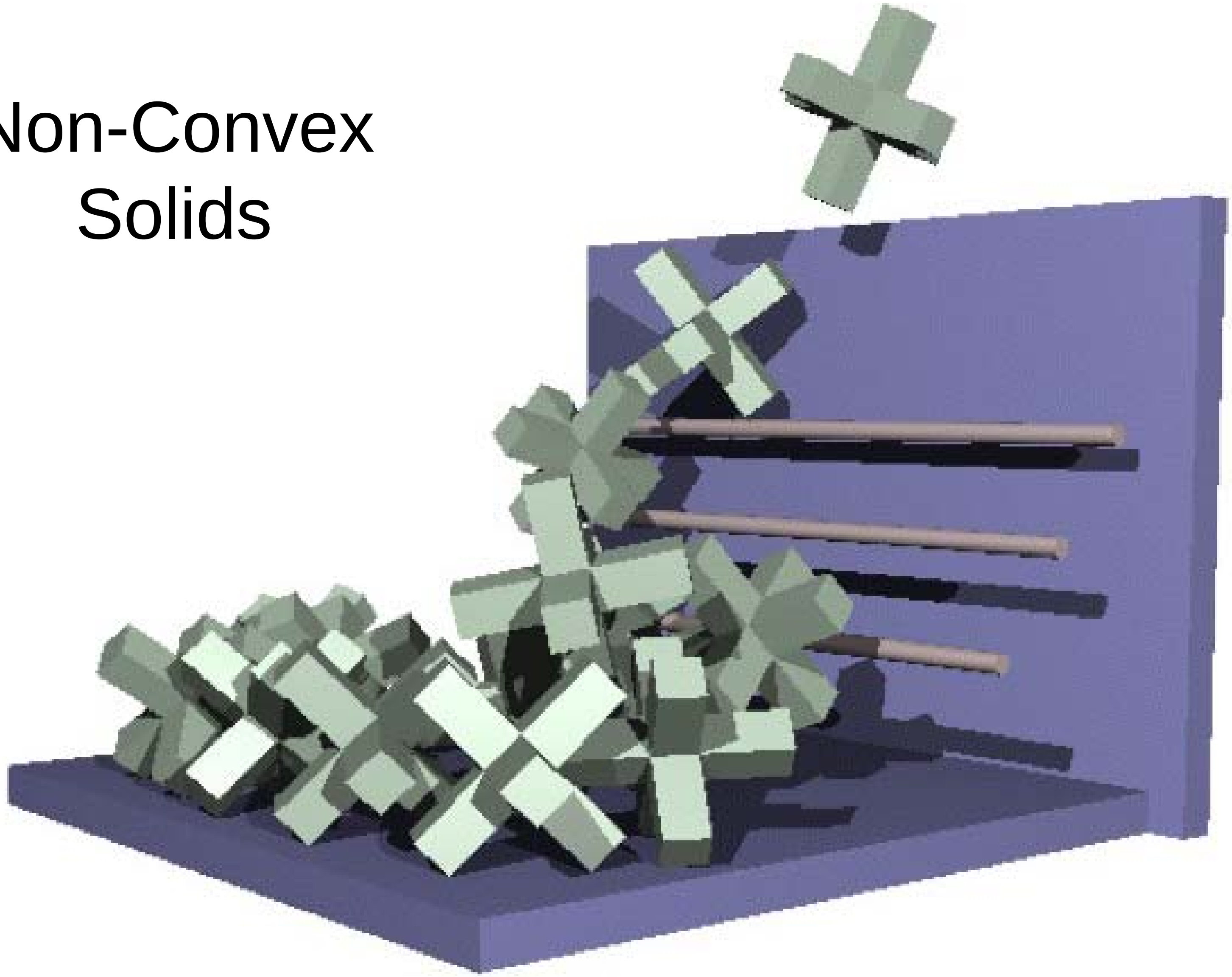

Figure 8: *Jacks*: an example of non-convexity.

Experiments and Results

	#solids	close pairs/fr.	collisions/fr.	contacts/fr.	#frames	sec./fr.	avg. #qp/fr.	rollbacks [%]
Stack	10	6.6	29	25.7	600	0.9	3	0
Cubejam	100	172	411.7	278.4	1500	22.2	4.8	0
Wall	90	129.2	404.3	220.6	600	12.3	4	0
Jacks	50(150)	123.3	167.1	94	1500	15.5	4.6	0.03
Pendulum	6	0	0	0	1000	0.1	2.1	0
Hybrid	100	159.1	518.7	497.6	1500	25.9	4.8	0.07
Hourglass	1000	1723.4	1673.6	1673.6	2000	13.5	2.6	0
Robot	306	308	507.7	507.7	580	75.5	6	2.7

Table 2: Complexity of scenes and efficiency issues.

Experiments and Results

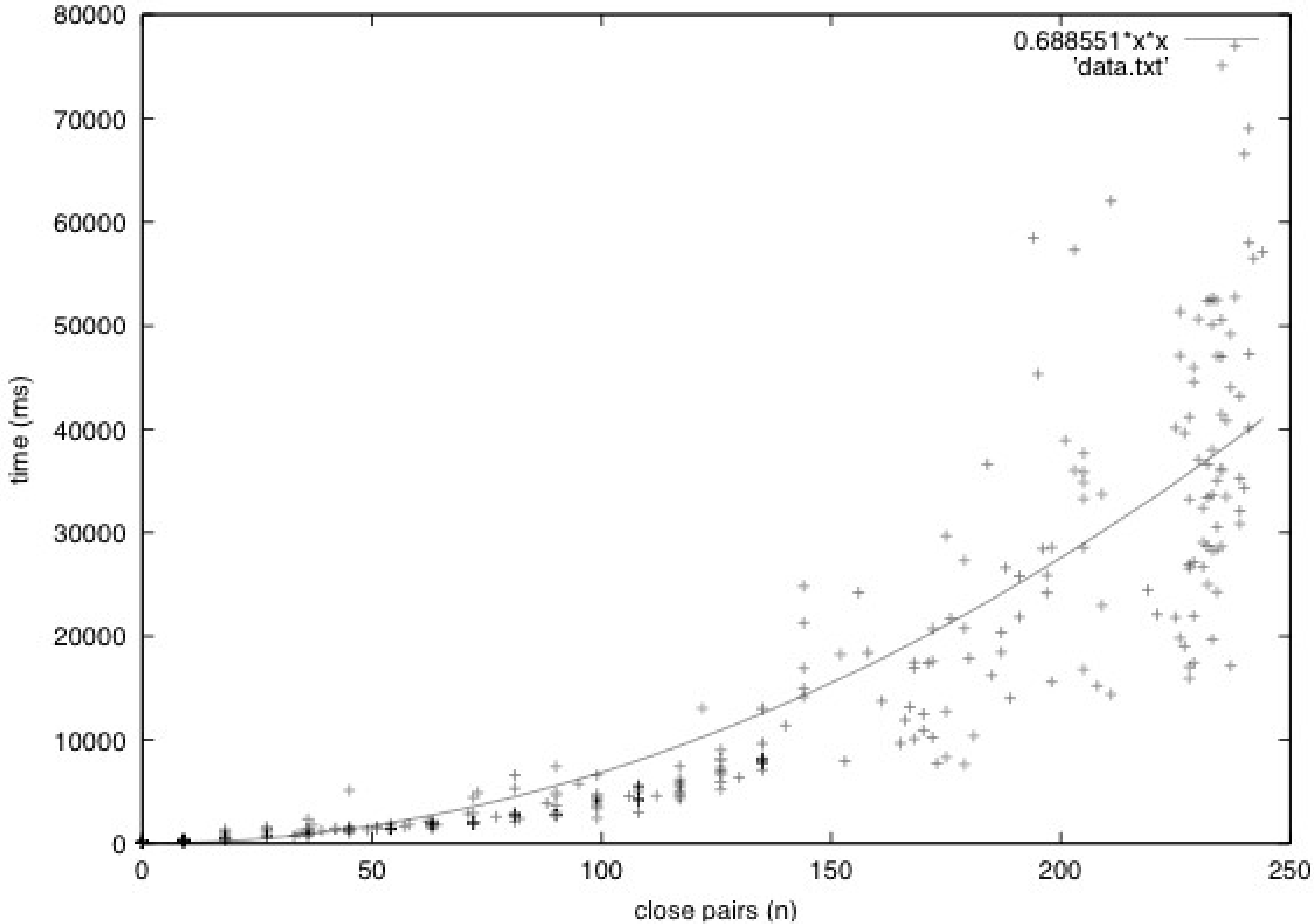

Figure 9: Solution time of one QP vs. number of close pairs for the position update.