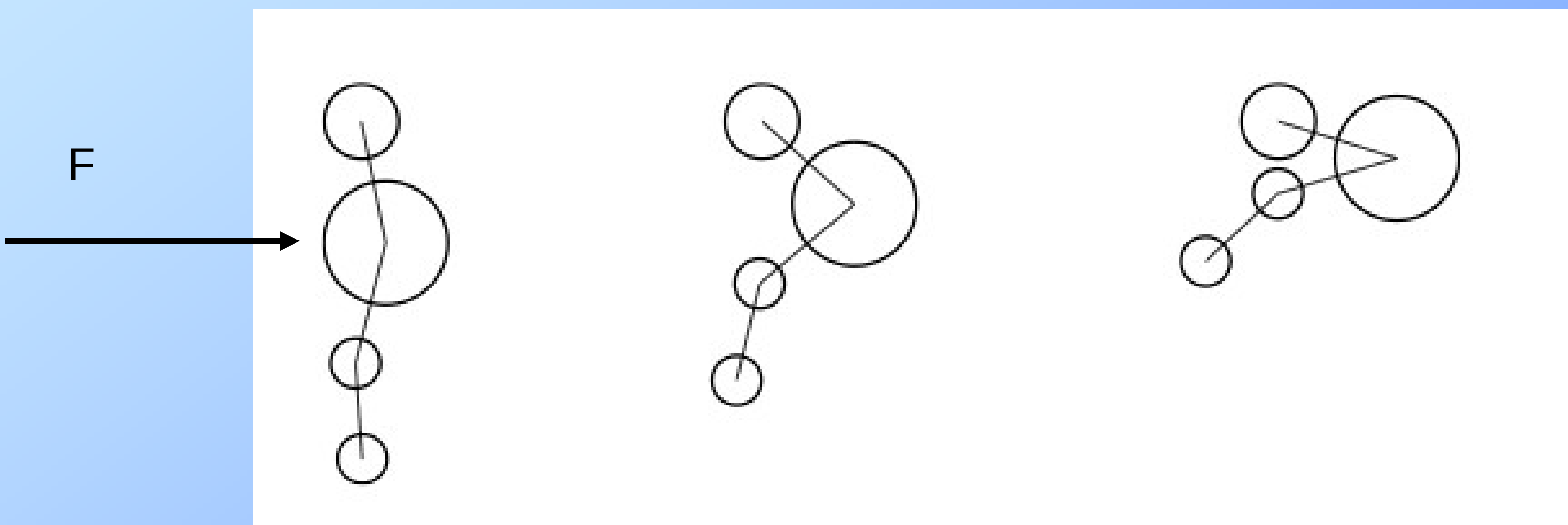
Dynamics of Linked Hierarchies

Constrained dynamics
The Featherstone equations

Constrained dynamics

Apply force to one component, other components repositioned, from near to far, to satisfy distance constraints



Constrained Body Dynamics

Chapter 4 in:

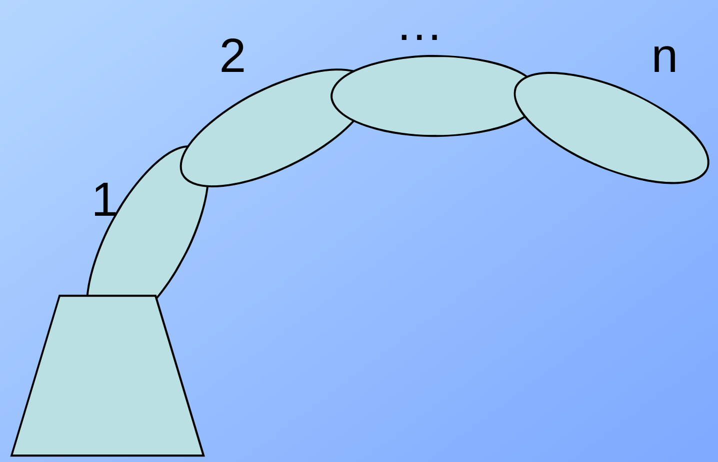
Mirtich

Impulse-based Dynamic Simulation of Rigid Body Systems

Ph.D. dissertation, Berkeley, 1996

Preliminaries

- Links numbered 0 to n
- Fixed base: link 0; Outermost like: link n
- Joints numbered 1 to n
- Link i has inboard joint, Joint i
- Each joint has 1 DoF
- Vector of joint positions: $q=(q_1,q_2,\ldots q_n)^T$



The Problem

- Given:
  - the **positions q and velocities** $\dot{q}$ of the n joints of a serial linkage,
  - the **external forces** acting on the linkage,
  - and the **forces and torques** being applied by the joint actuators
- Find: The resulting **accelerations** of the joints: $\ddot{q}$

First Determine equations that give absolute motion of all links

Given: the joint positions q, velocities and accelerations
Compute: for each link the linear and angular velocity and acceleration relative to an inertial frame

Notation – global variables

v_i	Linear velocity of link i
a_i	Linear acceleration of link i
ω_i	Angular velocity of link i
α_i	Angular acceleration of link i

Joint variables

q_i joint position

$\dot{q}_i$ joint velocity

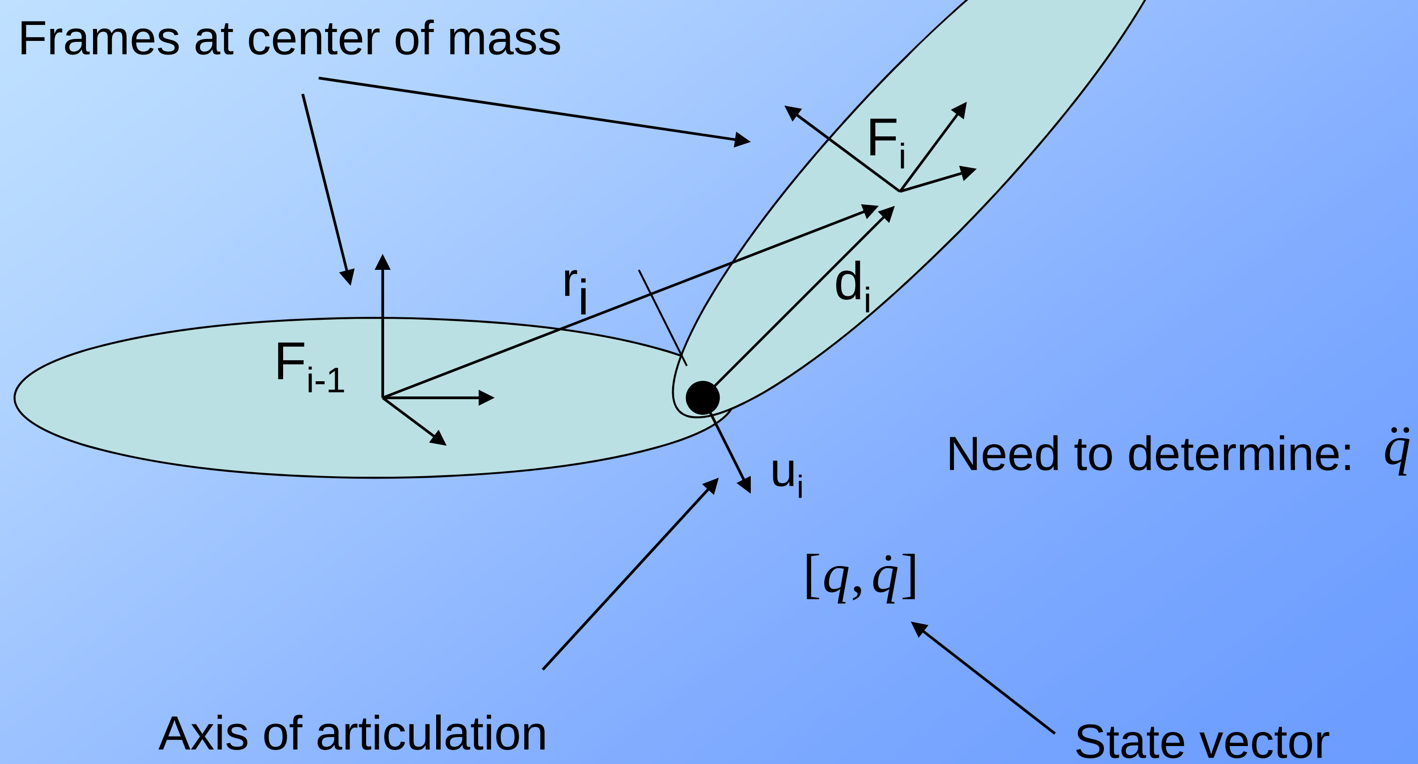
u_i Unit vector in direction of the axis of joint i

r_i vector from origin of F_{i-1} to origin of F_i

d_i vector from axis of joint i to origin of F_i

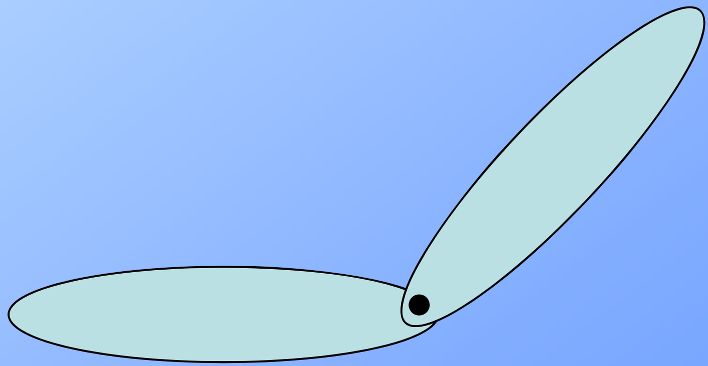
Basic terms

- F_i – body frame of link i
- Origin at center of mass
- Axes aligned with principle axes of inertia



From base outward

- Velocities and accelerations of link i are completely determined by:
  1. the velocities and accelerations of link i-1
  2. and the motion of joint i



First - determine velocities and accelerations

From velocity and acceleration of previous link, determine total (global) velocity and acceleration of current link

Computed from base outward

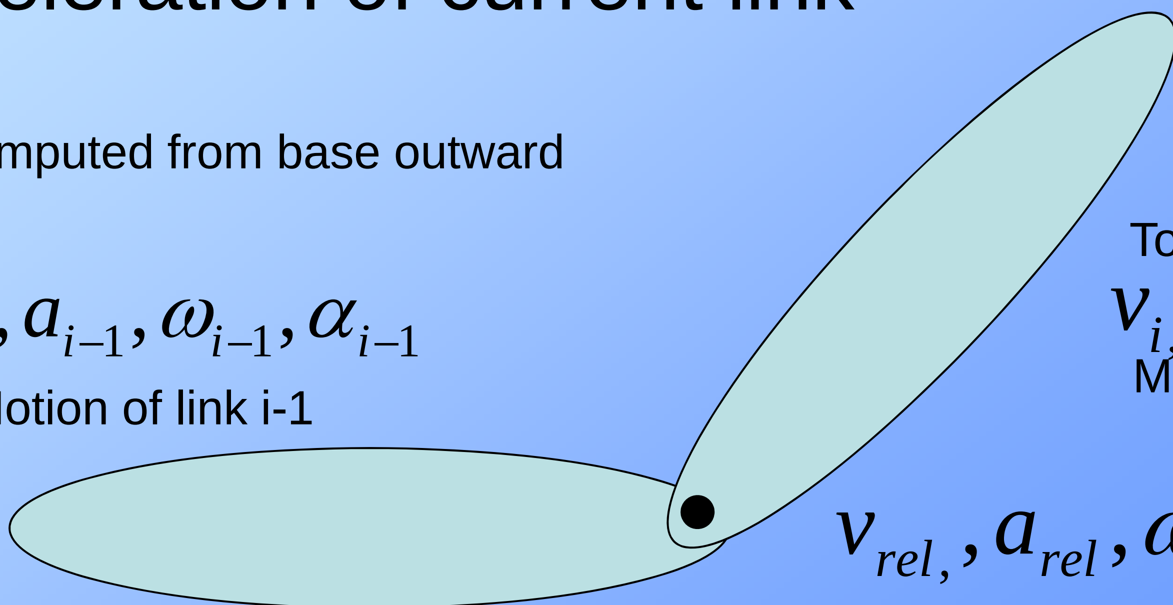
To be computed

$v_{i,}, a_i, \omega_i, \alpha_i$

Motion of link i

$v_{i-1,}, a_{i-1}, \omega_{i-1}, \alpha_{i-1}$

Motion of link i-1



$v_{rel,}, a_{rel}, \omega_{rel}, \alpha_{rel}$

Motion of link i from local joint

Compute outward

Angular velocity of link i =

- angular velocity of link i-1 plus
- angular velocity induced by rotation at joint i

$$\omega_i = \omega_{i-1} + \omega_{rel}$$

Linear velocity =

- linear velocity of link i-1 plus
- linear velocity induced by rotation at link -1 plus
- linear velocity from translation at joint i

$$v_i = v_{i-1} + \omega_{i-1} \times r_i + v_{rel}$$

Compute outward

Angular acceleration propagation $\alpha_i = \alpha_{i-1} + \dot{\omega}_{rel}$

Linear acceleration propagation

$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \omega_{i-1} \times \dot{r}_i + \dot{v}_{rel}$$

Rewritten, using $\dot{r}_i = v_i - v_{i-1}$ (relative velocity) and $v_i = v_{i-1} + \omega_{i-1} \times r_i + v_{rel}$ (from previous slide)

$$\dot{r}_i = \omega_{i-1} \times r_i + v_{rel}$$

$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \omega_{i-1} \times (\omega_{i-1} \times r_i) + \omega_{i-1} \times v_{rel} + \dot{v}_{rel}$$

Compute outward

Angular acceleration propagation

$$\alpha_i = \alpha_{i-1} + \dot{\omega}_{rel}$$

Linear acceleration propagation

$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \omega_{i-1} \times (\omega_{i-1} \times r_i) + \omega_{i-1} \times v_{rel} + \dot{v}_{rel}$$

Need $\begin{cases} \dot{\omega}_{rel} \\ \dot{v}_{rel} \end{cases}$

In terms of joint axis motion $\begin{cases} \dot{q}_i \\ \ddot{q}_i \\ u_i \end{cases}$

Define w_{rel} and v_{rel} and their time derivatives

Joint velocity vector

Axis times parametric velocity

$$\upsilon_i = \dot{q}_i u_i$$

Joint acceleration vector

Axis times parametric acceleration

$$\xi_i = \ddot{q}_i u_i$$ (unkown)

prismatic

$$\omega_{rel} = 0$$

$$v_{rel} = \upsilon_i$$

revolute

$$\omega_{rel} = \upsilon_i$$

$$v_{rel} = \upsilon_i \times d_i$$

Velocity propagation formulae

(revolute)

linear

$$v_i = v_{i-1} + \omega_{i-1} \times r_i + v_{rel}$$

$$v_{rel} = \upsilon_i \times d_i$$

$$v_i = v_{i-1} + \omega_{i-1} \times r_i + \upsilon_i \times d_i$$

angular

$$\omega_i = \omega_{i-1} + \omega_{rel}$$

$$\omega_{rel} = \upsilon_i$$

$$\omega_i = \omega_{i-1} + \upsilon_i$$

Time derivatives of v_{rel} and w_{rel}

(revolute)

Joint acceleration vector

Change in joint velocity vector

$$\dot{\omega}_{rel} = \xi_i + \omega_{i-1} \times \upsilon_i$$

$$\dot{v}_{rel} = 2\omega_{i-1} \times (\upsilon_i \times d_i) + \xi_i \times d_i + \upsilon_i \times (\upsilon_i \times d_i)$$

From joint acceleration vector

From change in joint velocity vector

From change in change in vector from joint to CoM

Derivation of $\dot{v}_{rel}$

(revolute)

$$\upsilon_i = \dot{q}_i u_i$$

$$\dot{\upsilon}_i = \ddot{q} u_i + \dot{q} \dot{u}_i = \xi_i + \dot{q}\dot{u} \qquad \dot{u}_i = \omega_{i-1} \times u_i$$

$$\dot{\upsilon}_i = \xi_i + \omega_{i-1} \times \upsilon_i$$

$$\frac{d}{dt}(v_{rel}) = \frac{d}{dt}(\upsilon_i \times d_i) = \dot{\upsilon}_i \times d_i + \upsilon_i \times \dot{d}_i$$

$$\dot{d}_i = \omega_i \times d_i = (\omega_{i-1} + \upsilon_i) \times d_i$$

$$\dot{v}_{rel} = 2\omega_{i-1} \times (\upsilon_i \times d_i) + \xi_i \times d_i + \upsilon_i \times (\upsilon_i \times d_i)$$

Acceleration propagation formulae

(revolute)

linear

$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \underline{\omega_{i-1} \times \dot{r}_i} + \underline{\dot{v}_{rel}}$$

Previously derived

$$\dot{v}_{rel} = 2\omega_{i-1} \times (\upsilon_i \times d_i) + \xi_i \times d_i + \upsilon_i \times (\upsilon_i \times d_i)$$

$$\dot{r}_i = \omega_{i-1} \times r_i + v_{rel}$$

$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \xi_i \times d_i + \omega_{i-1} \times (\omega_{i-1} \times r_i) + 2\omega_{i-1} \times (\upsilon_i \times d_i) + \upsilon_i \times (\upsilon_i \times d_i)$$

angular

$$\alpha_i = \alpha_{i-1} + \dot{\omega}_{rel}$$

$$\dot{\omega}_{rel} = \xi_i + \omega_{i-1} \times \upsilon_i$$

$$\alpha_i = \alpha_{i-1} + \xi_i + \omega_{i-1} \times \upsilon_i$$

Spatial formulation of acceleration propagation

(revolute)

$$v_i = v_{i-1} + \omega_{i-1} \times r_i + \upsilon_i \times d_i$$

$$\omega_i = \omega_{i-1} + \upsilon_i$$

$$\alpha_i = \alpha_{i-1} + \xi_i + \omega_{i-1} \times \upsilon_i$$

$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \xi_i \times d_i + \omega_{i-1} \times (\omega_{i-1} \times r_i) + 2\omega_{i-1} \times (\upsilon_i \times d_i) + \upsilon_i \times (\upsilon_i \times d_i)$$

But remember $\xi_i = \ddot{q}_i u_i$ is an unknown

First step in forward dynamics

- Use known dynamic state: $q, \dot{q}$
- Compute absolute linear and angular velocities: v, ω
- <u>Remember:</u> Acceleration propagation equations involve unknown joint accelerations

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But first – need to introduce notation to facilitate equation writing

Spatial Algebra

Spatial Algebra

Spatial velocity

$$\hat{v} = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

Spatial acceleration

$$\hat{a} = \begin{bmatrix} \alpha \\ a \end{bmatrix}$$

Spatial Transform Matrix

r – offset vector   R– rotation

$$\hat{v}_G = {}_G\hat{X}_F \hat{v}_F$$

$$\hat{a}_G = {}_G\hat{X}_F \hat{a}_F$$

$${}_G\hat{X}_F = \begin{bmatrix} R & 0 \\ -\tilde{r}R & R \end{bmatrix}$$

↑
(cross product operator)

Spatial Algebra

Spatial force

$$\hat{f} = \begin{bmatrix} f \\ \tau \end{bmatrix}$$

Spatial transpose

$$\hat{x}' = \begin{bmatrix} a \\ b \end{bmatrix}' = \begin{bmatrix} b^T & a^T \end{bmatrix}$$

Spatial joint axis

$$\hat{s}_i = \begin{bmatrix} u_i \\ u_i \times d_i \end{bmatrix}$$

(used in later)

Spatial inner product

$$\hat{x}'\hat{y}$$

ComputeSerialLinkVelocities

(revolute)

$$\omega_0, v_0, \alpha_0, a_0 \leftarrow 0$$

For i = 1 to N do

R←rotation matrix from frame i-1 to i

r←radius vector from frame i-1 to frame i (in frame i coordinates)

$$\omega_i \leftarrow R\omega_{i-1}$$

$$v_i \leftarrow Rv_{i-1} + \omega_i \times r$$

$$\left.\begin{aligned} \omega_i &\leftarrow \omega_i + \dot{q}_i u_i \\ v_i &\leftarrow v_i + \dot{q}_i (u_i \times d_i) \end{aligned}\right\} \text{Specific to revolute joints}$$

end

Spatial formulation of acceleration propagation

Previously:

(revolute)

$$\alpha_i = \alpha_{i-1} + \xi_i + \omega_{i-1} \times \upsilon_i$$

$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \xi_i \times d_i + \omega_{i-1} \times (\omega_{i-1} \times r_i) + 2\omega_{i-1} \times (\upsilon_i \times d_i) + \upsilon_i \times (\upsilon_i \times d_i)$$

Want to put in form:

$$\widehat{a}_i = {}_i\widehat{X}_{i-1}\widehat{a}_{i-1} + \ddot{q}_i \widehat{s} + \widehat{c}_i$$

Where: $${}_G\widehat{X}_F = \begin{bmatrix} R & 0 \\ -\tilde{r}R & R \end{bmatrix} \qquad \widehat{s}_i = \begin{bmatrix} u_i \\ u_i \times d_i \end{bmatrix}$$

Spatial Coriolis force

(revolute)

$$\alpha_i = \alpha_{i-1} + \xi_i + \omega_{i-1} \times \upsilon_i$$

$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \xi_i \times d_i + \omega_{i-1} \times (\omega_{i-1} \times r_i) + 2\omega_{i-1} \times (\upsilon_i \times d_i) + \upsilon_i \times (\upsilon_i \times d_i)$$

$$\hat{a}_i = {}_i\hat{X}_{i-1}\hat{a}_{i-1} + \ddot{q}_i \hat{s}_i + \hat{c}_i$$

$$\hat{s}_i = \begin{bmatrix} u_i \\ u_i \times d_i \end{bmatrix}$$

These are the terms involving $\xi_i = \ddot{q}_i u_i$

$$\hat{c}_i \leftarrow \begin{bmatrix} \omega_{i-1} \times \upsilon_i \\ \omega_{i-1} \times (\omega_{i-1} \times r_i) + 2\omega_{i-1} \times (\upsilon_i \times d_i) + \upsilon_i \times (\upsilon_i \times d_i) \end{bmatrix}$$

Featherstone algorithm

$\hat{a}_i$ Spatial acceleration of link i

$\hat{f}_i^{\,I}$ Spatial force exerted on link i through its inboard joint

$\hat{f}_i^{\,O}$ Spatial force exerted on link i through its outboard joint

All expressed in frame i

Forces expressed as acting on center of mass of link i

Serial linkage articulated motion

$$\widehat{f}_i^{\,I} = \widehat{I}_i^{\,A}\widehat{a}_i + \widehat{Z}_i^{\,A}$$

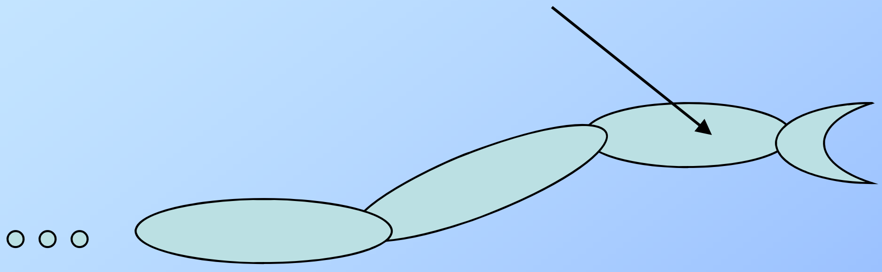
$\widehat{I}_i^{\,A}$ Spatial articulated inertia of link I; *articulated* means entire subchain is being considered

$\widehat{Z}_i^{\,A}$ Spatial articulated zero acceleration force of link I (independent of joint accelerations); force exerted by inboard joint on link i, if link i is not to accelerate

Develop equations by induction

Base Case

Consider last link of linkage (link n)



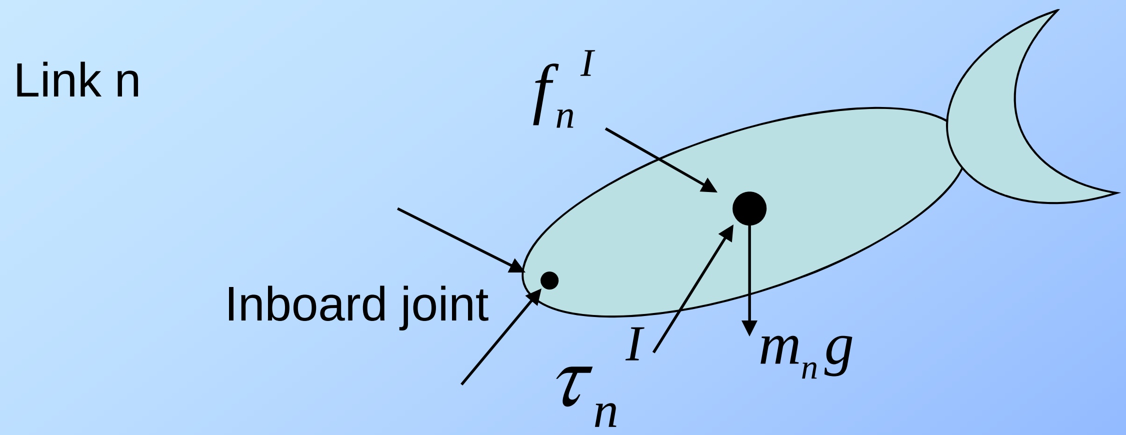
Force/torque applied by inboard joint + gravity = inertia*accelerations of link

Newton-Euler equations of motion

$$\widehat{f}_n^{\,I} + m_n g = m_n a_n$$

$$\tau_n^{\,I} = I_n \alpha_n + \omega_n \times I_n \omega_n$$

Using spatial notation

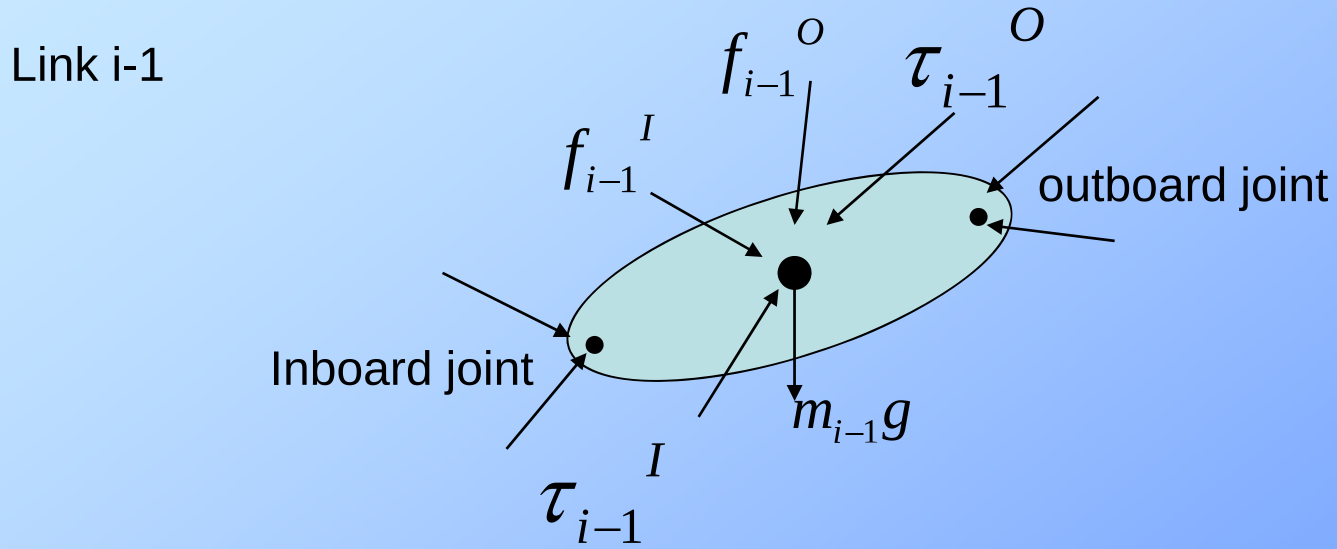


$$\begin{bmatrix} f_n^{\ I} \\ \tau_n^{\ I} \end{bmatrix} = \begin{bmatrix} 0 & M_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} \alpha_n \\ a_n \end{bmatrix} + \begin{bmatrix} -m_n g \\ \omega_n \times I_n \omega_n \end{bmatrix}$$

$$\widehat{f}_n = \widehat{I}_n^{\ A} \widehat{a}_n + \widehat{Z}_n^{\ A}$$

Inductive case

Assume previous is true for link i; consider link i-1



$$\begin{bmatrix} f_{i-1}^{I} \\ \tau_{i-1}^{I} \end{bmatrix} + \begin{bmatrix} f_{i-1}^{O} \\ \tau_{i-1}^{O} \end{bmatrix} = \begin{bmatrix} 0 & M_{i-1} \\ I_{i-1} & 0 \end{bmatrix} \begin{bmatrix} \alpha_{i-1} \\ a_{i-1} \end{bmatrix} + \begin{bmatrix} -m_{i-1}g \\ \omega_{i-1} \times I_{i-1}\omega_{i-1} \end{bmatrix}$$

Inductive case

$$\begin{bmatrix} f_{i-1}{}^{I} \\ \tau_{i-1}{}^{I} \end{bmatrix} + \begin{bmatrix} f_{i-1}{}^{O} \\ \tau_{i-1}{}^{O} \end{bmatrix} = \begin{bmatrix} 0 & M_{i-1} \\ I_{i-1} & 0 \end{bmatrix} \begin{bmatrix} \alpha_{i-1} \\ a_{i-1} \end{bmatrix} + \begin{bmatrix} -m_{i-1} g \\ \omega_{i-1} \times I_{i-1} \omega_{i-1} \end{bmatrix}$$

$$\widehat{f}_{i-1}{}^{I} = \widehat{I}_{i-1} \widehat{a}_{i-1} + \widehat{Z}_{i-1} - \widehat{f}_{i-1}{}^{O}$$

The effect of joint I on link i-1 is equal and opposite to its effect on link i

$$\widehat{f}_{i-1}{}^{O} = -{}_{i-1}X_{i} \widehat{f}_{i}{}^{I}$$

Substituting…

$$\widehat{f}_{i-1}{}^{I} = \widehat{I}_{i-1} \widehat{a}_{i-1} + \widehat{Z}_{i-1} + {}_{i-1}X_{i} \widehat{f}_{i}{}^{I}$$

Inductive case

$$\hat{f}_{i-1}{}^{I} = \hat{I}_{i-1}\hat{a}_{i-1} + \hat{Z}_{i-1} + {}_{i-1}X_{i}\,\hat{f}_{i}{}^{I}$$

Invoking induction on the definition of $\hat{f}_{i}{}^{I}$

$$\hat{f}_{i-1}{}^{I} = \hat{I}_{i-1}\hat{a}_{i-1} + \hat{Z}_{i-1} + {}_{i-1}X_{i}\,(\hat{I}_{i}{}^{A}\hat{a}_{i} + \hat{Z}_{i}{}^{A})$$

Inductive case

$$\widehat{f}_{i-1}^{\,I} = \widehat{I}_{i-1}\widehat{a}_{i-1} + \widehat{Z}_{i-1} + {}_{i-1}X_i(\widehat{I}_i^{\,A}\widehat{a}_i + \widehat{Z}_i^{\,A})$$

Express a_i in terms of a_{i-1} and rearrange

$$\widehat{a}_i = {}_i\widehat{X}_{i-1}\widehat{a}_{i-1} + \ddot{q}_i\widehat{s}_i + \widehat{c}_i$$

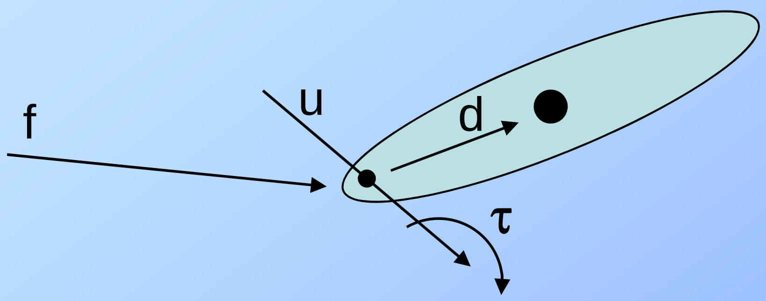
$$\widehat{f}_{i-1}^{\,I} = (\widehat{I}_{i-1} + {}_{i-1}X_i\widehat{I}_i^{\,A}\,{}_iX_{i-1})\widehat{a}_{i-1} + \widehat{Z}_{i-1} + {}_{i-1}X_i[\widehat{Z}_i^{\,A} + \widehat{I}_i^{\,A}\widehat{c}_i + (\widehat{I}_i^{\,A}\widehat{s}_i)\ddot{q}_i]$$

Need to eliminate from the right side of the equation $\ddot{q}_i$

Inductive case

$$\hat{f}_{i-1}^{\ I} = (\hat{I}_{i-1} + {}_{i-1}X_i \hat{I}_i^{\ A}\ {}_iX_{i-1})\hat{a}_{i-1} + \hat{Z}_{i-1} + {}_{i-1}X_i[\hat{Z}_i^{\ A} + \hat{I}_i^{\ A}\hat{c}_i + (\hat{I}_i^{\ A}\hat{s}_i)\ddot{q}_i]$$

Magnitude of torque exerted by revolute joint actuator is Q_i



A force f and a torque τ applied to link i at the inboard joint give rise to a spatial inboard force (resolved in the body frame) of $\hat{f}_i^{\ I} = \begin{bmatrix} f_i \\ \tau_i - d_i \times f_i \end{bmatrix}$

$$Q_i = \hat{s}_i{}' \hat{f}_i^{\ I} = \begin{bmatrix} u_i \\ u_i \times d_i \end{bmatrix}' \begin{bmatrix} f_i \\ \tau_i - d_i \times f_i \end{bmatrix} = f_i \cdot (u_i \times d_i) + (\tau_i - d_i \times f_i) \cdot u_i = \tau_i \cdot u_i$$

Moment of force

Moment of force

Inductive case

$$Q_i = \hat{s}_i{}' \hat{f}_i{}^I = \tau_i \cdot u_i$$

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previously $\hat{a}_i = {}_i\hat{X}_{i-1}\hat{a}_{i-1} + \ddot{q}_i \hat{s}_i + \hat{c}_i$ and $\hat{f}_i{}^I = \hat{I}_i{}^A \hat{a}_i + \hat{Z}_i{}^A$

$$\hat{f}_i{}^I = \hat{I}_i{}^A ({}_i\hat{X}_{i-1}\hat{a}_{i-1} + \ddot{q}_i \hat{s}_i + \hat{c}_i) + \hat{Z}_i{}^A$$

Premultiply both sides by $\hat{s}_i{}'$ substitute Q_i for s’f , and solve

$$\ddot{q}_i = \frac{Q_i - \hat{s}_i{}' \hat{I}_i{}^A {}_i\hat{X}_{i-1}\hat{a}_{i-1} - \hat{s}_i{}' (\hat{Z}_i{}^A + \hat{I}_i{}^A \hat{c}_i)}{\hat{s}_i{}' \hat{I}_i{}^A \hat{s}_i}$$

And substitute

$$\ddot{q}_i = \frac{Q_i - \hat{s}_i' \hat{I}_i^A \, {}_i\hat{X}_{i-1} \hat{a}_{i-1} - \hat{s}_i' (\hat{Z}_i^A + \hat{I}_i^A \hat{c}_i)}{\hat{s}_i' \hat{I}_i^A \hat{s}_i}$$

$$\hat{f}_{i-1}^I = (\hat{I}_{i-1} + {}_{i-1}X_i \hat{I}_i^A \, {}_iX_{i-1}) \hat{a}_{i-1} + \hat{Z}_{i-1} + {}_{i-1}X_i [\hat{Z}_i^A + \hat{I}_i^A \hat{c}_i + (\hat{I}_i^A \hat{s}_i) \ddot{q}_i]$$

$$\hat{f}_{i-1}^I = [\hat{I}_{i-1} + {}_{i-1}X_i (\hat{I}_i^A - \frac{\hat{I}_i^A \hat{s}_i \hat{s}_i' \hat{I}_i^A}{\hat{s}_i' \hat{I}_i^A \hat{s}_i})_i X_{i-1}] \hat{a}_{i-1} +$$

$$\hat{Z}_{i-1} + {}_{i-1}X_i [\hat{Z}_i^A + \hat{I}_i^A \hat{c}_i + \frac{\hat{I}_i^A \hat{s}_i [Q_i - \hat{s}_i' \hat{I}_i^A \, {}_i\hat{X}_{i-1} \hat{a}_{i-1} - \hat{s}_i' (\hat{Z}_i^A + \hat{I}_i^A \hat{c}_i)]}{\hat{s}_i' \hat{I}_i^A \hat{s}_i}]$$

And form I & Z terms

$$\hat{f}_{i-1}^{\,I} = [\hat{I}_{i-1} + {}_{i-1}X_i(\hat{I}_i^{\,A} - \frac{\hat{I}_i^{\,A}\hat{s}_i\hat{s}_i'\hat{I}_i^{\,A}}{\hat{s}_i'\hat{I}_i^{\,A}\hat{s}_i})_i X_{i-1}]\hat{a}_{i-1} +$$

$$\hat{Z}_{i-1} + {}_{i-1}X_i[\hat{Z}_i^{\,A} + \hat{I}_i^{\,A}\hat{c}_i + \frac{\hat{I}_i^{\,A}\hat{s}_i[Q_i - \hat{s}_i'\hat{I}_i^{\,A}\,{}_i\hat{X}_{i-1}\hat{a}_{i-1} - \hat{s}_i'(\hat{Z}_i^{\,A} + \hat{I}_i^{\,A}\hat{c}_i)]}{\hat{s}_i'\hat{I}_i^{\,A}\hat{s}_i}]$$

To get into form: $\hat{f}_i^{\,I} = \hat{I}_i^{\,A}\hat{a}_i + \hat{Z}_i^{\,A}$

$$\hat{I}_{i-1}^{\,A} = \hat{I}_{i-1} + {}_{i-1}X_i(\hat{I}_i^{\,A} - \frac{\hat{I}_i^{\,A}\hat{s}_i\hat{s}_i'\hat{I}_i^{\,A}}{\hat{s}_i'\hat{I}_i^{\,A}\hat{s}_i})_i X_{i-1}$$

$$\hat{Z}_{i-1}^{\,A} = \hat{Z}_{i-1} + {}_{i-1}X_i[\hat{Z}_i^{\,A} + \hat{I}_i^{\,A}\hat{c}_i + \frac{\hat{I}_i^{\,A}\hat{s}_i[Q_i - \hat{s}_i'\hat{I}_i^{\,A}\,{}_i\hat{X}_{i-1}\hat{a}_{i-1} - \hat{s}_i'(\hat{Z}_i^{\,A} + \hat{I}_i^{\,A}\hat{c}_i)]}{\hat{s}_i'\hat{I}_i^{\,A}\hat{s}_i}]$$

Ready to put into code

Using

- Loop from inside out to compute velocities previously developed (repeated on next slide)
- Loop from inside out to initialize I, Z, and c variables
- Loop from outside in to propagate I, Z and c updates
- Loop from inside out to compute $\ddot{q}$ using I, Z, c

ComputeSerialLinkVelocities

(revolute)

// This is code from an earlier slide – loop inside out to compute velocities

$$\omega_0, v_0, \alpha_0, a_0 \leftarrow 0$$

For i = 1 to N do

R←rotation matrix from frame i-1 to i

r←radius vector from frame i-1 to frame i (in frame i coordinates)

$$\omega_i \leftarrow R\omega_{i-1}$$

$$v_i \leftarrow Rv_{i-1} + \omega_i \times r$$

$$\left.\begin{aligned} \omega_i &\leftarrow \omega_i + \dot{q}_i u_i \\ v_i &\leftarrow v_i + \dot{q}_i (u_i \times d_i) \end{aligned}\right\} \quad \text{Specific to revolute joints}$$

end

InitSerialLinks

(revolute)

// loop from inside out to initialize Z, I, c variables

For i = 1 to N do

$$\widehat{Z}_i^{\,A} \leftarrow \begin{bmatrix} -m_i g \\ \omega_i \times I_i \omega_i \end{bmatrix}$$

$$\widehat{I}_i^{\,A} \leftarrow \begin{bmatrix} 0 & M_i \\ I_i & 0 \end{bmatrix}$$

$$\widehat{c}_i \leftarrow \begin{bmatrix} \omega_{i-1} \times \upsilon_i \\ \omega_{i-1} \times (\omega_{i-1} \times r_i) + 2\omega_{i-1} \times (\upsilon_i \times d_i) + \upsilon_i \times (\upsilon_i \times d_i) \end{bmatrix}$$

end

SerialForwardDynamics

// new code with calls to 2 previous routines

Call compSerialLinkVelocities

Call initSerialLinks

// loop outside in to form I and Z for each linke

For i = n to 2 do

$$\hat{I}_{i-1}^{A} \leftarrow \hat{I}_{i-1}^{A} + {}_{i-1}\hat{X}_{i}[\hat{I}_{i}^{A} - \frac{\hat{I}_{i}^{A}\hat{s}_{i}\hat{s}_{i}'\hat{I}_{i}^{A}}{\hat{s}_{i}'\hat{I}_{i}^{A}\hat{s}_{i}}]_{i}\hat{X}_{i-1}$$

$$\hat{Z}_{i-1}^{A} \leftarrow \hat{Z}_{i-1}^{A} + {}_{i-1}\hat{X}_{i}[\hat{Z}_{i}^{A} + \hat{I}_{i}^{A}\hat{c}_{i} + \frac{\hat{I}_{i}^{A}\hat{s}_{i}[Q_{i} - \hat{s}_{i}'(\hat{Z}_{i}^{A} + \hat{I}^{A}{}_{i}\hat{c}_{i})]}{\hat{s}_{i}'\hat{I}_{i}^{A}\hat{s}_{i}}]$$

// loop inside out to compute link and joint accelerations

$$\hat{a}_{0} \leftarrow 0$$

For i = 1 to n do

$$\ddot{q}_{i} = \frac{Q_{i} - \hat{s}_{i}'\hat{I}_{i}^{A}{}_{i}\hat{X}_{i-1}\hat{a}_{i-1} - \hat{s}_{i}'(\hat{Z}_{i}^{A} + \hat{I}_{i}^{A}\hat{c}_{i})}{\hat{s}_{i}'\hat{I}_{i}^{A}\hat{s}_{i}}$$

$$\hat{a}_{i} = {}_{i}\hat{X}_{i-1}\hat{a}_{i-1} + \hat{c}_{i} + \ddot{q}_{i}\hat{s}_{i}$$

And that's all there is to it!

$$\omega_i = \omega_{i-1} + \omega_{rel}$$
$$v_i = v_{i-1} + \omega_{i-1} \times r_i + v_{rel}$$
$$\alpha_i = \alpha_{i-1} + \dot{\omega}_{rel}$$
$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \omega_{i-1} \times \dot{r}_i + \dot{v}_{rel}$$
$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \omega_{i-1} \times (\omega_{i-1} \times r_i) + \omega_{i-1} \times v_{rel} + \dot{v}_{rel}$$
$$\dot{\omega}_{rel} = \xi_i + \omega_{i-1} \times \upsilon_i$$
$$\dot{v}_{rel} = \omega_{i-1} \times (\upsilon_i \times d_i) + \xi_i \times d_i + \upsilon_i \times (\upsilon_i \times d_i)$$
$$\frac{d}{dt}(v_{rel}) = \frac{d}{dt}(\upsilon_i \times d_i) = \dot{\upsilon}_i \times d_i + \upsilon_i \times \dot{d}_i$$
$$\dot{d}_i = \omega_i \times d_i = (\omega_{i-1} + \upsilon_i) \times d_i$$
$$\dot{v}_{rel} = 2\omega_{i-1} \times (\upsilon_i \times d_i) + \xi_i \times d_i + \upsilon_i \times (\upsilon_i \times d_i)$$

$$\dot{r}_i = v_i - v_{i-1}$$
$$\dot{r}_i = \omega_{i-1} \times r_i + v_{rel}$$

$$\upsilon_i = \dot{q}_i u_i \qquad \omega_{rel} = \upsilon_i$$
$$\xi_i = \ddot{q}_i u_i \qquad v_{rel} = \upsilon_i \times d_i$$

$$\dot{\upsilon}_i = \xi_i + \omega_{i-1} \times \upsilon_i$$

$$v_i = v_{i-1} + \omega_{i-1} \times r_i + \upsilon_i \times d_i$$
$$\omega_i = \omega_{i-1} + \upsilon_i$$

$$a_i = a_{i-1} + \alpha_{i-1} \times r_i + \xi_i \times d_i + \omega_{i-1} \times (\omega_{i-1} \times r_i) + 2\omega_{i-1} \times (\upsilon_i \times d_i) + \upsilon_i \times (\upsilon_i \times d_i)$$

$$\alpha_i = \alpha_{i-1} + \xi_i + \omega_{i-1} \times \upsilon_i$$

$$\hat{v} = \begin{bmatrix} \omega \\ v \end{bmatrix} \quad \hat{a} = \begin{bmatrix} \alpha \\ a \end{bmatrix} \qquad {}_G\hat{X}_F = \begin{bmatrix} R & 0 \\ -\tilde{r}R & R \end{bmatrix}$$

$$\hat{f} = \begin{bmatrix} f \\ \tau \end{bmatrix} \qquad \hat{x}' = \begin{bmatrix} a \\ b \end{bmatrix}' = \begin{bmatrix} b^T & a^T \end{bmatrix}$$

$$\hat{s}_i = \begin{bmatrix} u_i \\ u_i \times d_i \end{bmatrix} \qquad \hat{x}'\hat{y}$$