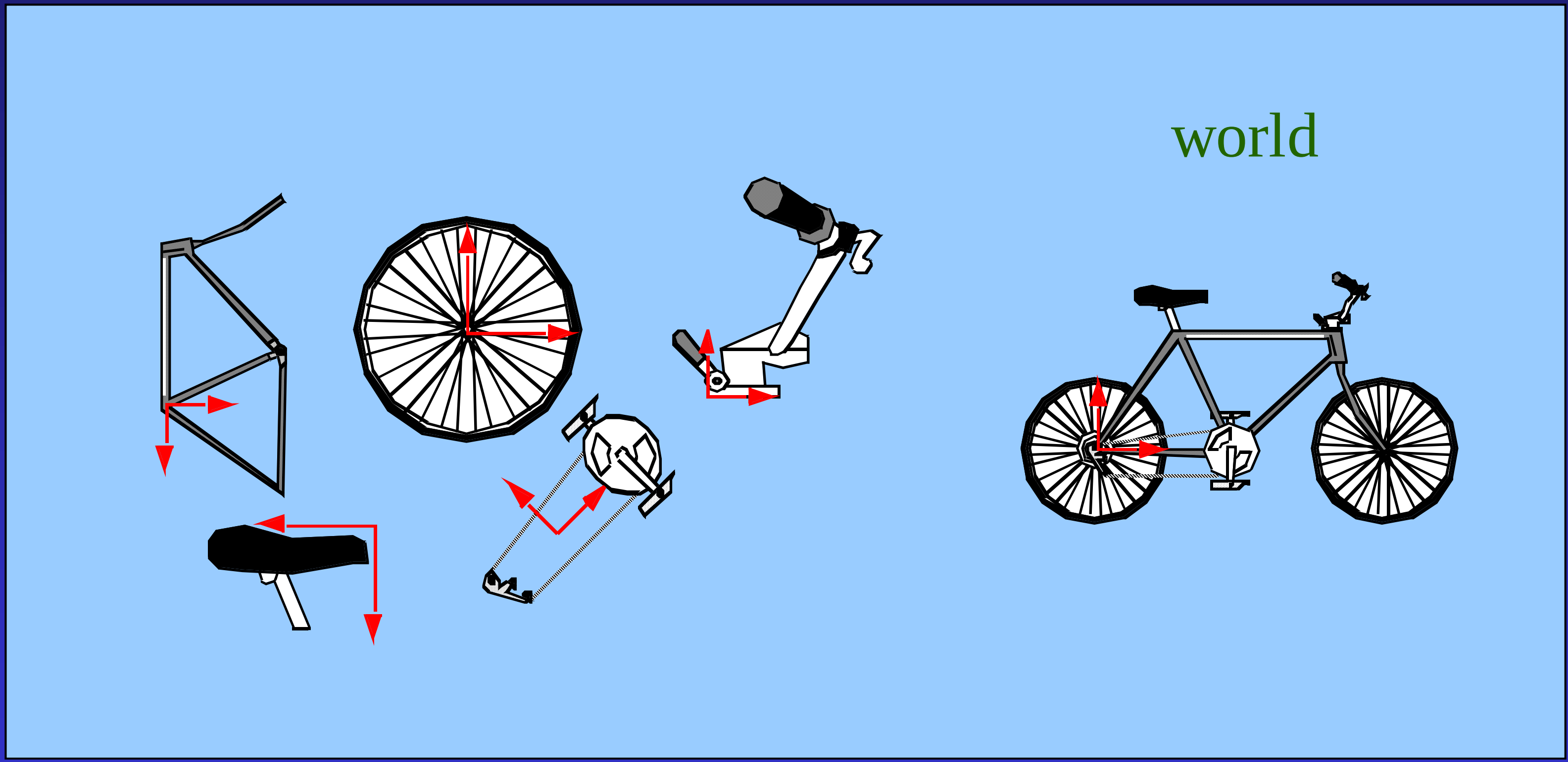
Review: Transformations

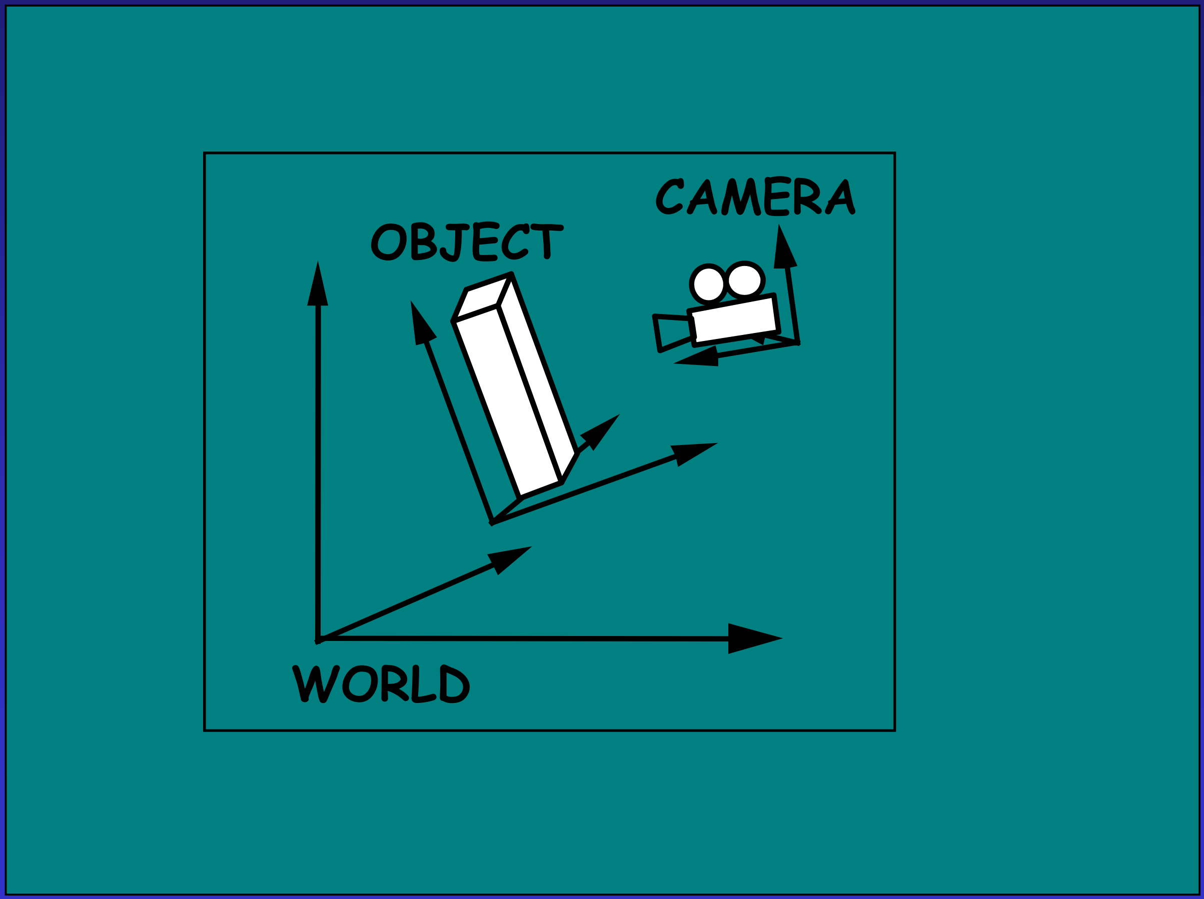
Transformations

- Modeling transformations
  - build complex models by positioning (transforming) simple components relative to each other
- Viewing transformations
  - placing virtual camera in the world
  - transformation from world coordinates to camera coordinates
  - Perspective projection of 3D coordinates to 2D
- Animation
  - vary transformations over time to create motion

Transformations - Modeling

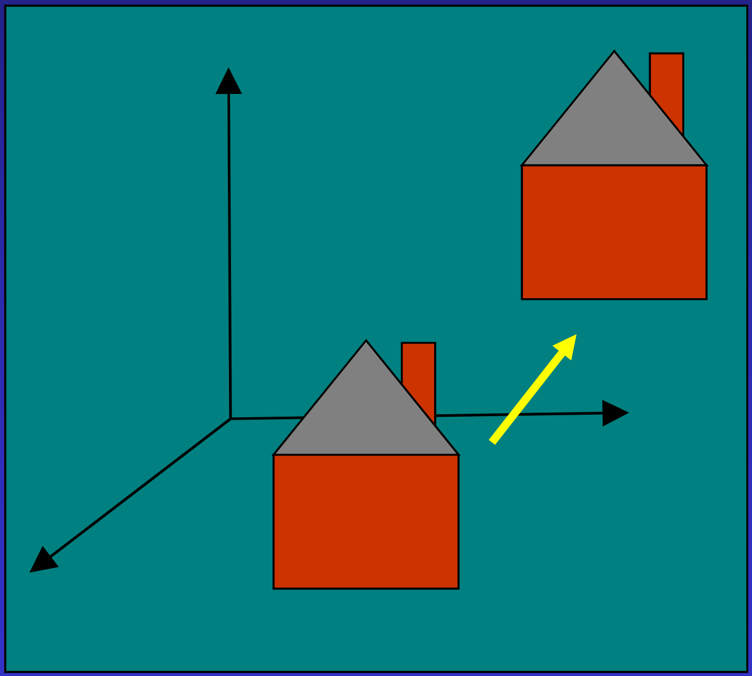


Transformations - Viewing

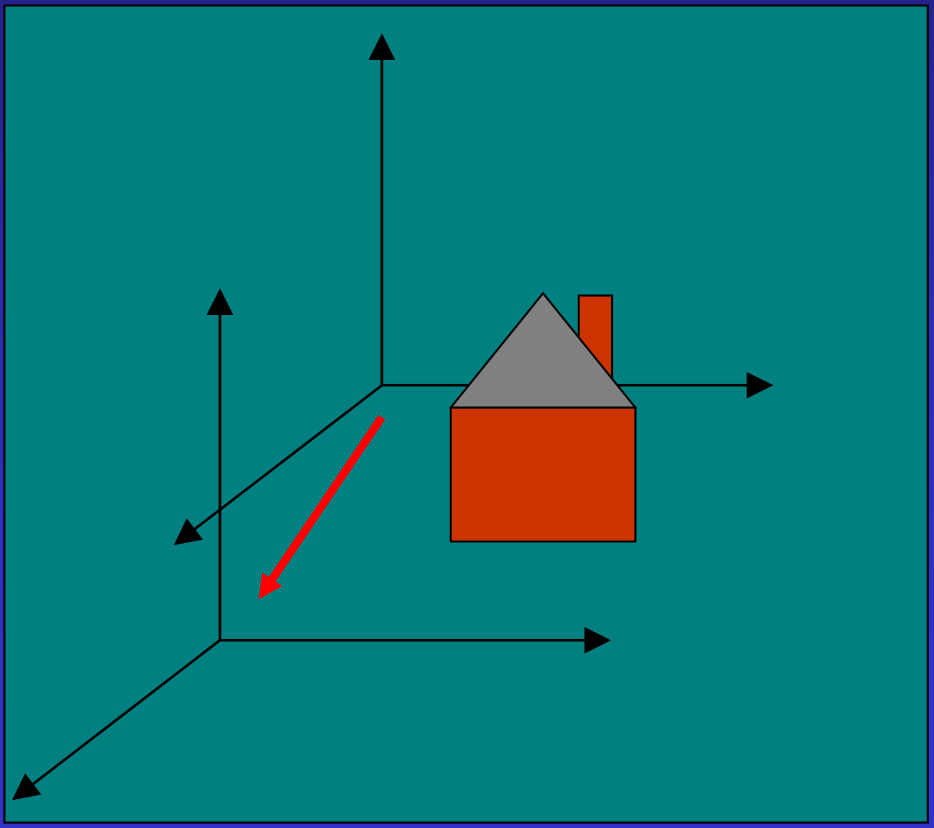


Modeling Transformations

Transform objects/points



Transform coordinate system



Affine Transformations

Transform P = (x, y, z) to Q = (x', y', z') by M.

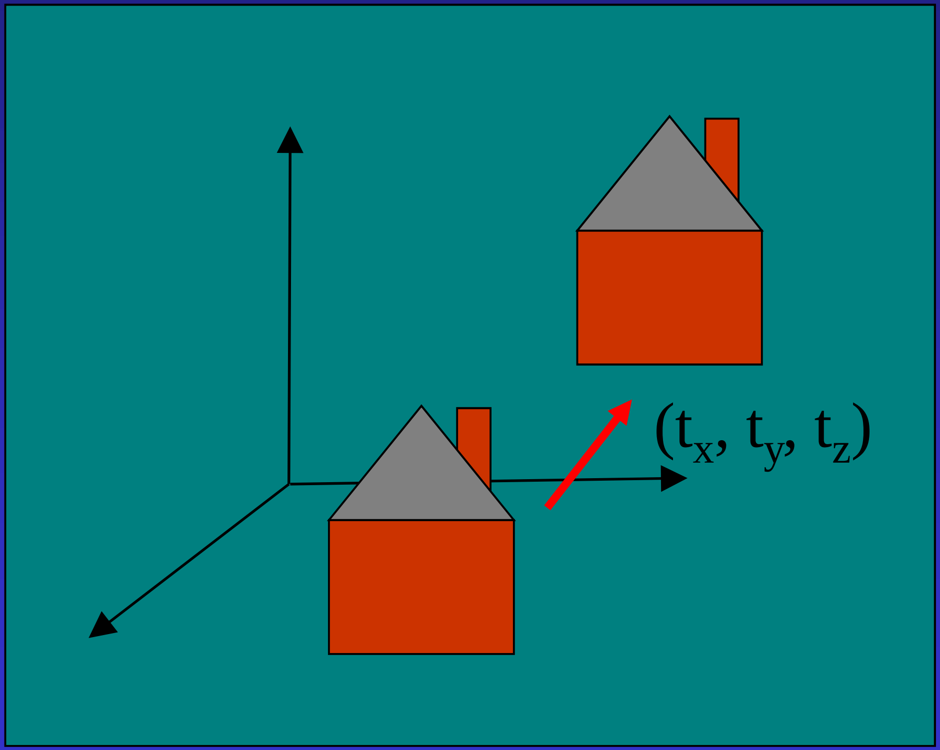
Affine transformation:

$$x' = m_{11}x + m_{12}y + m_{13}z + m_{14}$$
$$y' = m_{21}x + m_{22}y + m_{23}z + m_{24}$$
$$z' = m_{31}x + m_{32}y + m_{33}z + m_{34}$$

Translation

Translation by (t_x, t_y, t_z):

$$x' = x + t_x$$
$$y' = y + t_y$$
$$z' = z + t_z$$



Scaling

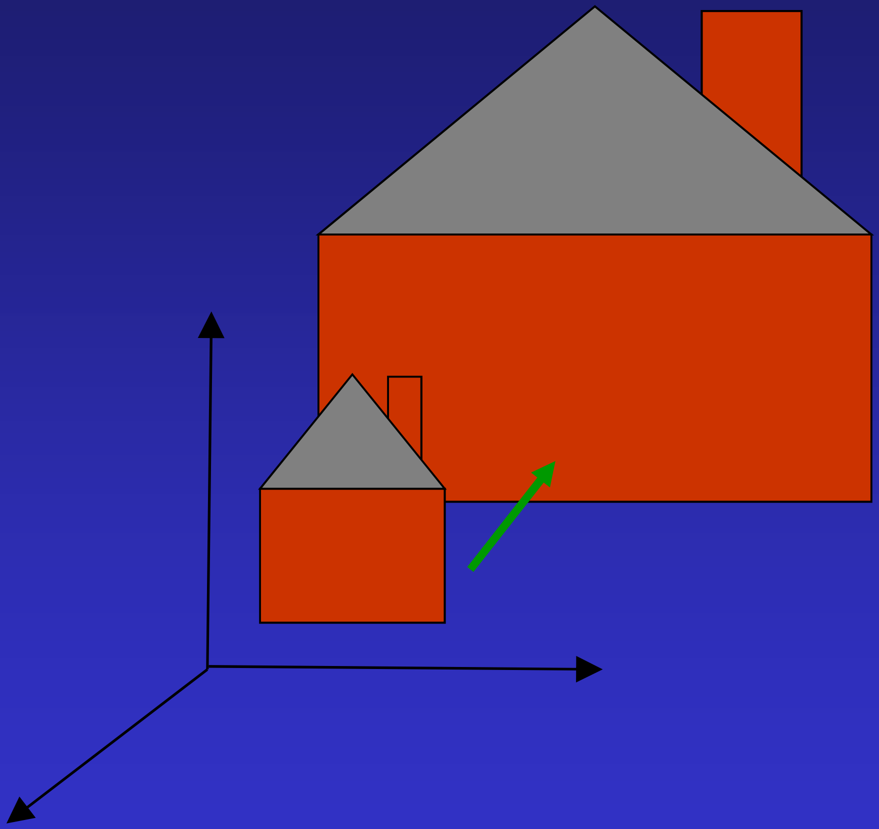
Scaling by (s_x, s_y, s_z):

$$x' = s_x x$$

$$y' = s_y y$$

$$z' = s_z z$$

Uniform v. non-uniform scaling



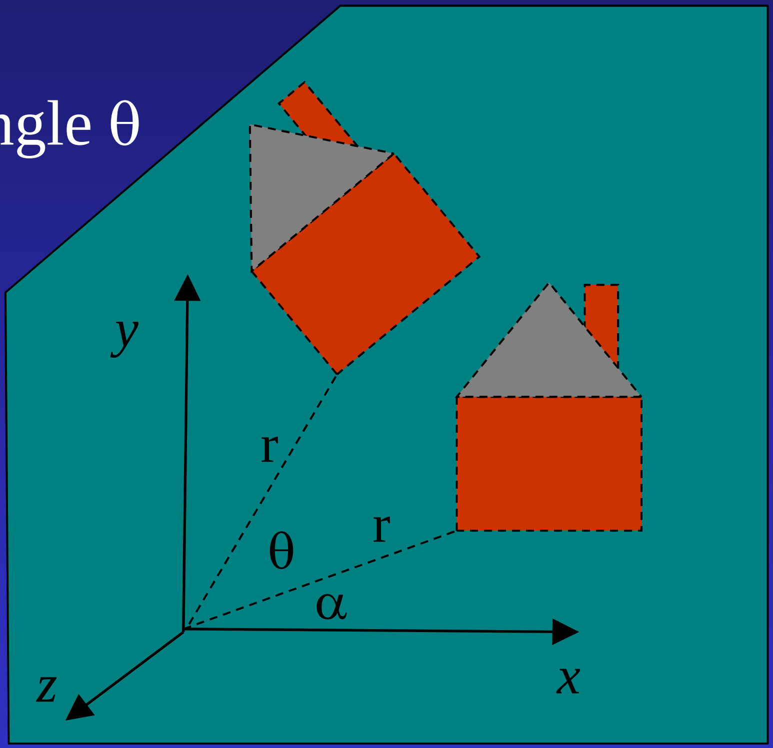
Rotation

Rotation counter-clockwise by angle θ around the *z*-axis:

$x' = x \cos(\theta) - y \sin(\theta)$

$y' = x \sin(\theta) + y \cos(\theta)$

$z' = z$



Proof:

$x = r \cos(\alpha)$

$y = r \sin(\alpha)$

$x' = r \cos(\alpha + \theta) = r \cos(\alpha) \cos(\theta) - r \sin(\alpha) \sin(\theta) = x \cos(\theta) - y \sin(\theta)$

$y' = r \sin(\alpha + \theta) = r \cos(\alpha) \sin(\theta) + r \sin(\alpha) \cos(\theta) = x \sin(\theta) + y \cos(\theta)$

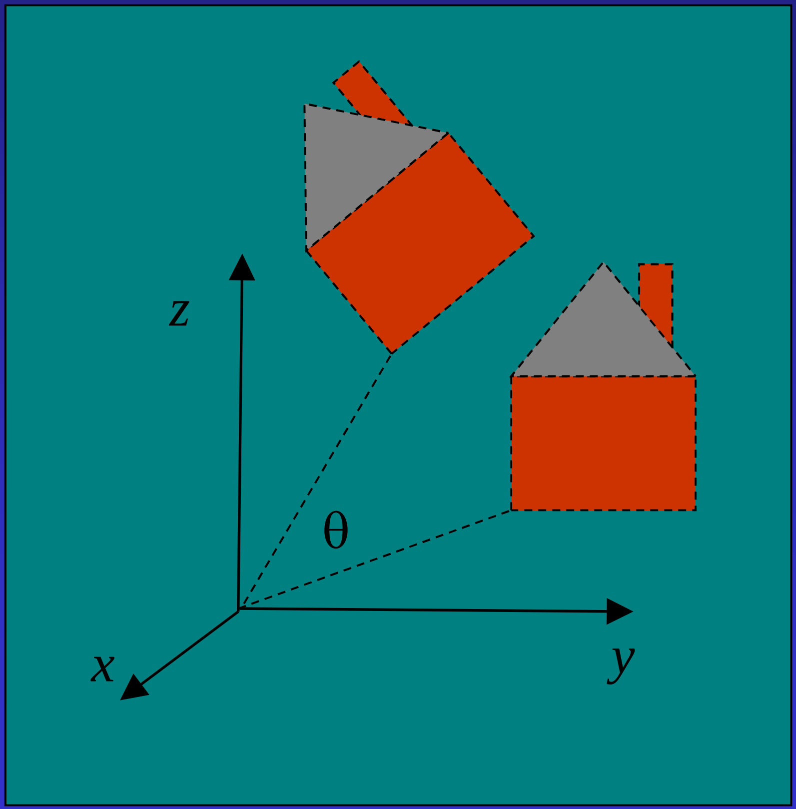
Rotation around *x*-axis

Rotation counter-clockwise by angle θ around the *x*-axis:

$x' = x$

$y' = y \cos(\theta) - z \sin(\theta)$

$z' = y \sin(\theta) + z \cos(\theta)$



Rotation around *y*-axis

Rotation counter-clockwise by angle θ around the *y*-axis:

$y' = y$

$z' = z\cos(\theta) - x\sin(\theta)$

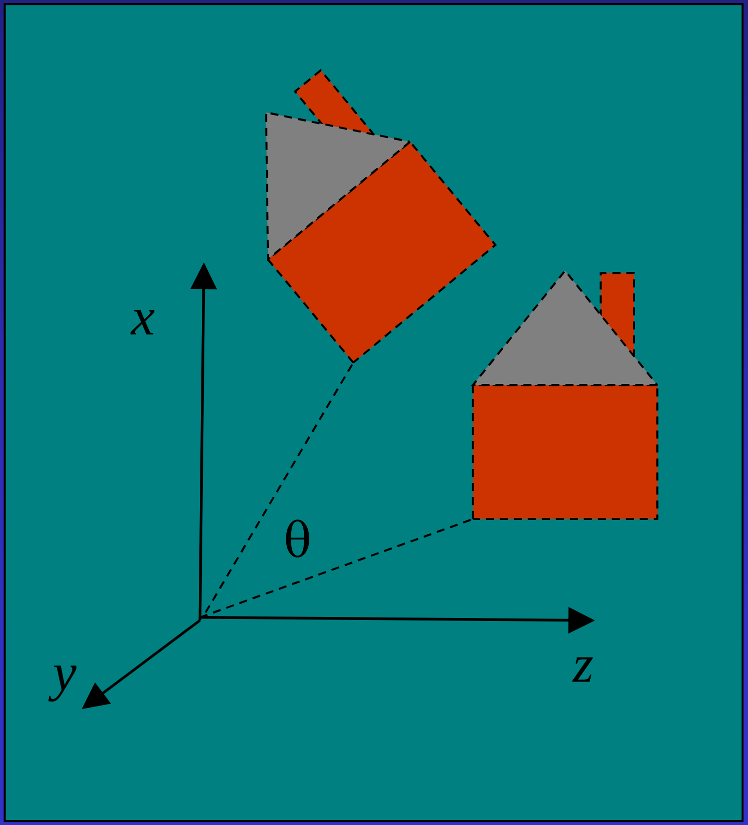
$x' = z\sin(\theta) + x\cos(\theta)$

Or

$x' = x\cos(\theta) + z\sin(\theta)$

$y' = y$

$z' = -x\sin(\theta) + z\cos(\theta)$



Matrix Multiplication

Scaling by (s_x, s_y, s_z):

$x' = s_x\, x$

$y' = s_y\, y$

$z' = s_z\, z$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Rotation counter-clockwise by angle θ around the *z*-axis:

$x' = x \cos(\theta) - y \sin(\theta)$

$y' = x \sin(\theta) + y \cos(\theta)$

$z' = z$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Translation by (t_x, t_y, t_z):

$x' = x + t_x$

$y' = y + t_y$

$z' = z + t_z$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Homogeneous Coordinates

Represent P by (x, y, z, 1) and Q by (x', y', z', 1).

(Homogeneous coordinates.)

Translation by (t_x, t_y, t_z):

$x' = x + t_x$

$y' = y + t_y$

$z' = z + t_z$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Scaling by (s_x, s_y, s_z):

$x' = s_x\, x$

$y' = s_y\, y$

$z' = s_z\, z$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Rotation Matrices

Rotation counter-clockwise by angle θ around the *x*-axis:

x' = x

y' = y cos(θ) – z sin(θ)

z' = y sin(θ) + z cos(θ)

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Rotation counter-clockwise by angle θ around the *y*-axis:

x' = x cos(θ) + z sin(θ)

y' = y

z' = – x sin(θ) + z cos(θ)

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Rotation counter-clockwise by angle θ around the *z*-axis:

x' = x cos(θ) – y sin(θ)

y' = x sin(θ) + y cos(θ)

z' = z

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Affine Transformation Matrix

Affine transformation:

$x' = m_{11}\,x + m_{12}\,y + m_{13}\,z + m_{14}$

$y' = m_{21}\,x + m_{22}\,y + m_{23}\,z + m_{24}$

$z' = m_{31}\,x + m_{32}\,y + m_{33}\,z + m_{34}$

Transformation Matrix:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Shearing

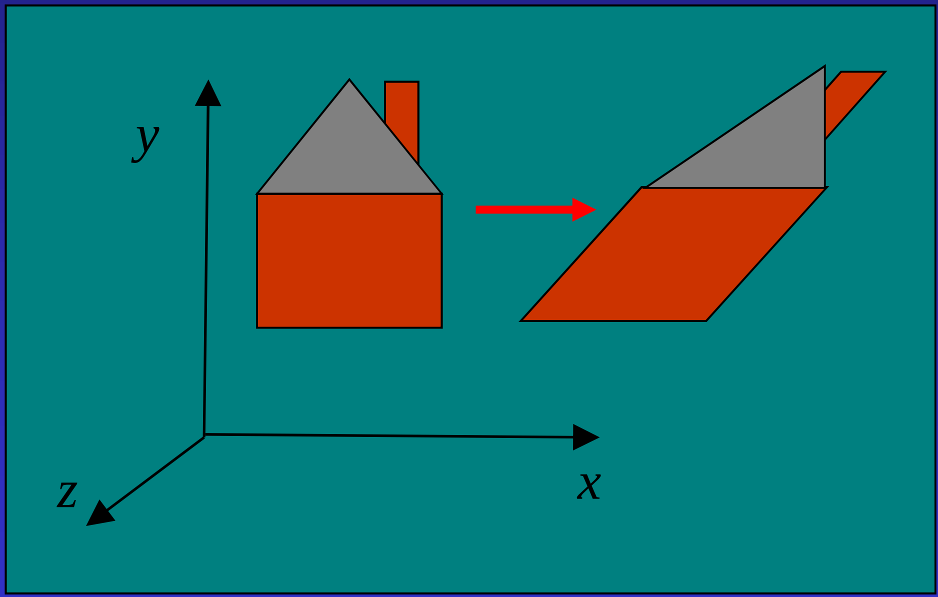
Shear along the *x*-axis:

x’ = x + hy

y’ = y

z’ = z

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$



Reflection

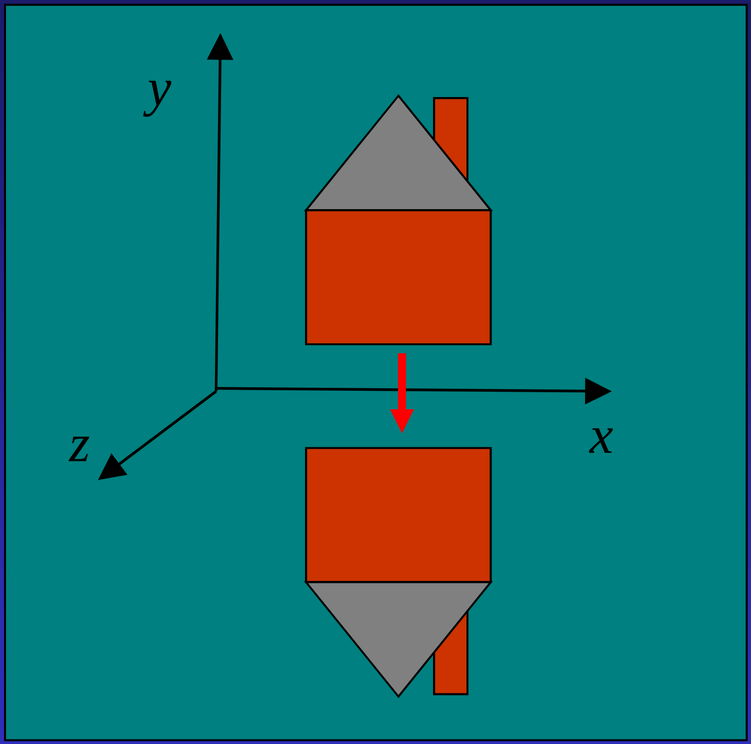
Reflection across the *x*-axis:

x' = (-1) x

y' = y

z' = z

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$



Reflection is a special case of scaling!

Elementary Transformations

- Translation
- Rotation
- Scaling
- Shear

What about inverses?

Inverse Transformations

Translation

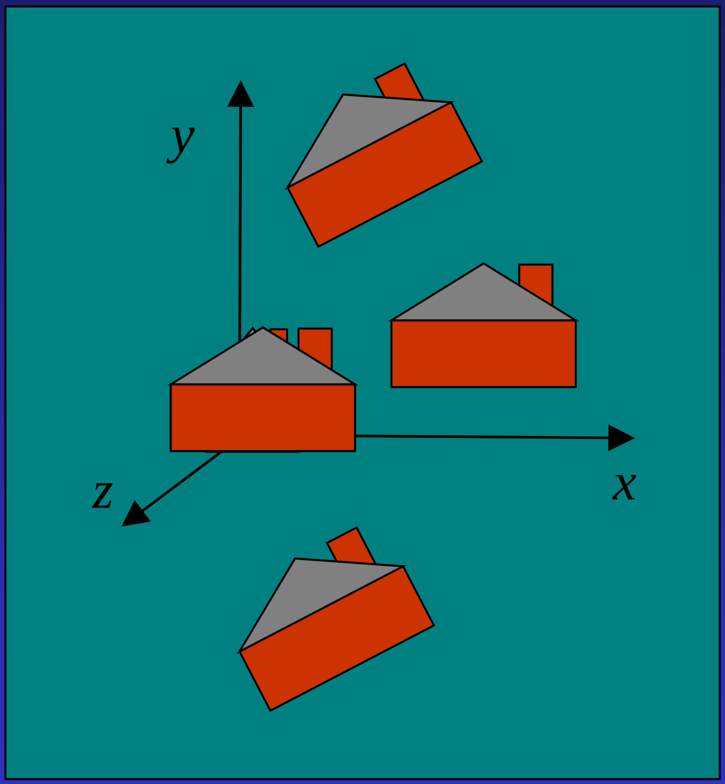
Scale

Rotation

Shear

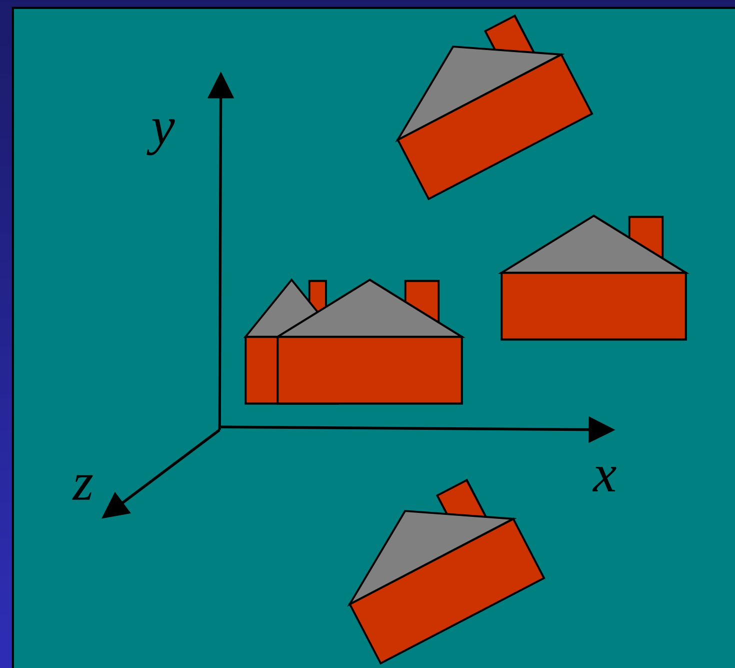
Compose Transformations

- Scale by (2, 1, 1)
- Translate by (20, 5, 0)
- Rotate by 30° counter-clockwise around the *z*-axis
- Translate by (0, -50, 0)



Compose Transformation Matrices

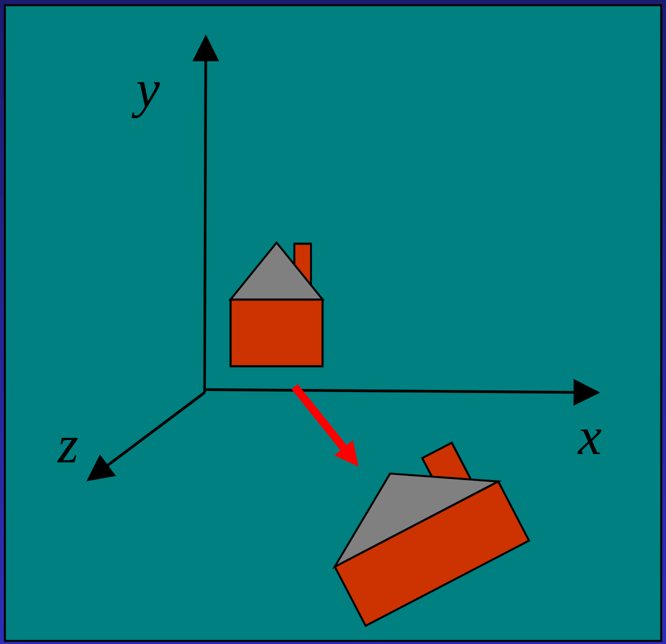
- Scale by (2, 1, 1);
- Translate by (20, 5, 0);
- Rotate by 30° counter-clockwise around the *z*-axis;
- Translate by (0, -50, 0).



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -50 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Translate}} \underbrace{\begin{bmatrix} 0.86 & -0.5 & 0 & 0 \\ 0.5 & 0.86 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Rotate}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Translate}} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Scale}} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Compose Transformation Matrices

- Scale by (2, 1, 1);
- Translate by (20, 5, 0);
- Rotate by 30° counter-clockwise around the *z*-axis;
- Translate by (0, -50, 0).

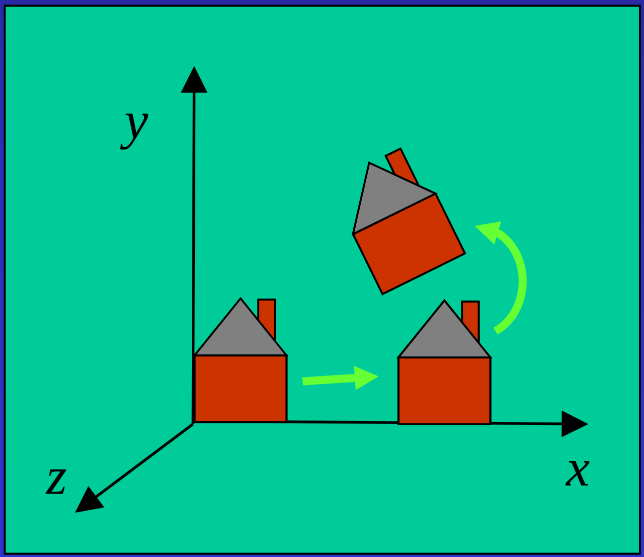


$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} 1.72 & -0.5 & 0 & 14.7 \\ 1.0 & 0.88 & 0 & 44.3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

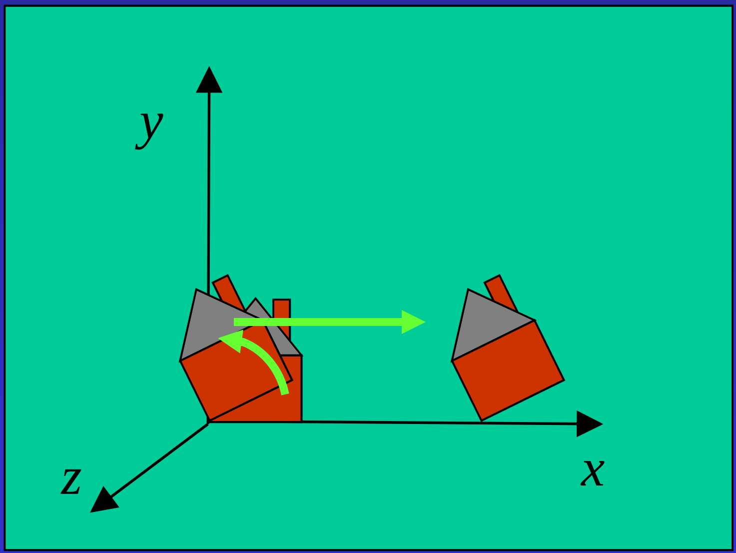
Order Matters!

Transformations are not necessarily commutative!

- Translate by (20, 0, 0).
- Rotate by 30°.



- Rotate by 30°.
- Translate by (20, 0, 0).



Order of Transformation Matrices

Apply transformation matrices from right to left.

- Translate by (20, 0, 0).
- Rotate by 30°.

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} 0.86 & -0.5 & 0 & 0 \\ 0.5 & 0.86 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

- Rotate by 30°.
- Translate by (20, 0, 0).

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.86 & -0.5 & 0 & 0 \\ 0.5 & 0.86 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Which transformations commute?

- Translation and translation?
- Scaling and scaling?
- Rotation and rotation?
- Translation and scaling?
- Translation and rotation?
- Scaling and rotation?

Elementary Transformations

Elementary transformations:

- Translation;
- Rotation;
- Scaling;
- Shear.

Theorem: Every affine transformation can be decomposed into elementary operations.

Euler's Theorem: Every rotation around the origin can be decomposed into a rotation around the *x*-axis followed by a rotation around the *y*-axis followed by a rotation around the *z*-axis. (or a single rotation about an arbitrary axis).

Vectors

Vector V = (V_x, V_y, V_z).

Homogeneous coordinates: $(V_x, V_y, V_z, 0)$.

Affine transformation:

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \\ 0 \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \\ 0 \end{pmatrix}.$$

Vectors

Rotation:

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \\ 0 \end{pmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \\ 0 \end{pmatrix}.$$

Scaling:

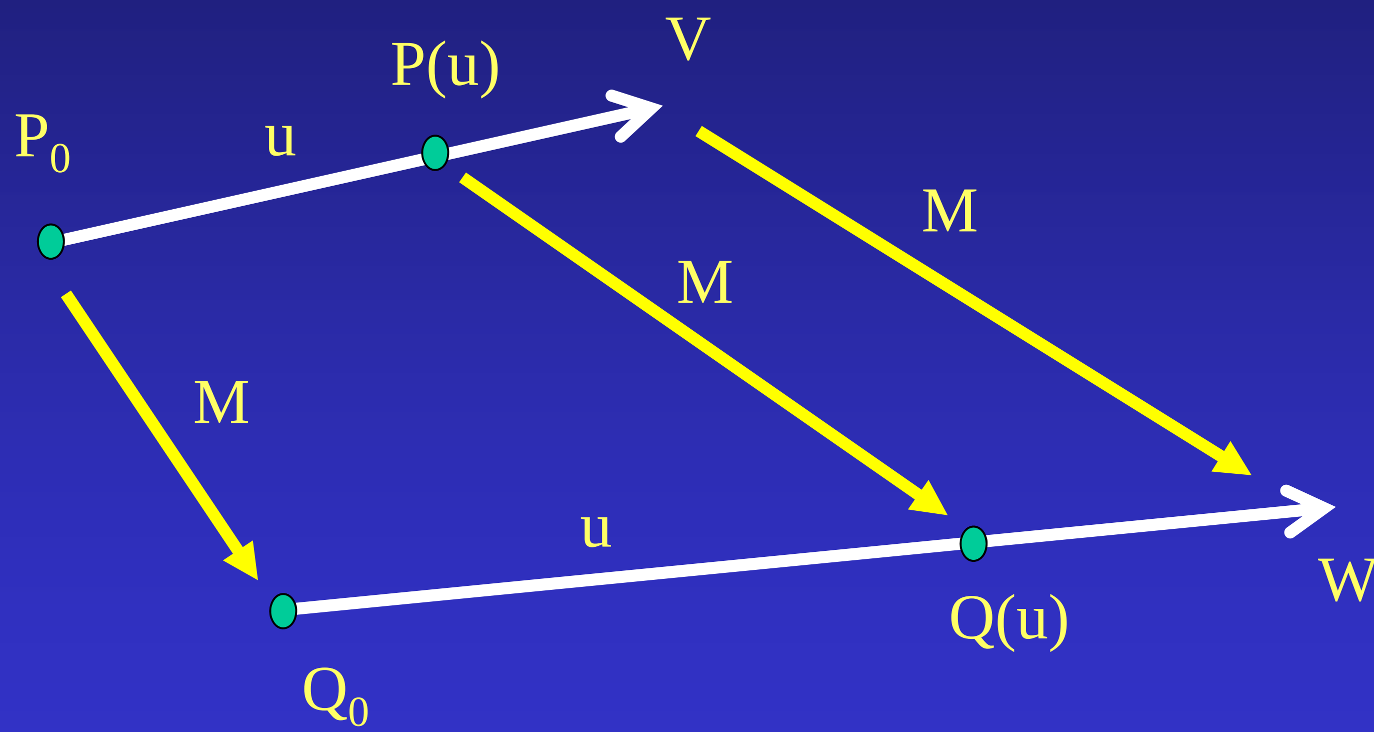
$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \\ 0 \end{pmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \\ 0 \end{pmatrix}.$$

Translation:

(Note: No change.)

$$\begin{pmatrix} V_x \\ V_y \\ V_z \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \\ 0 \end{pmatrix}.$$

Transform Parametric Line



Line

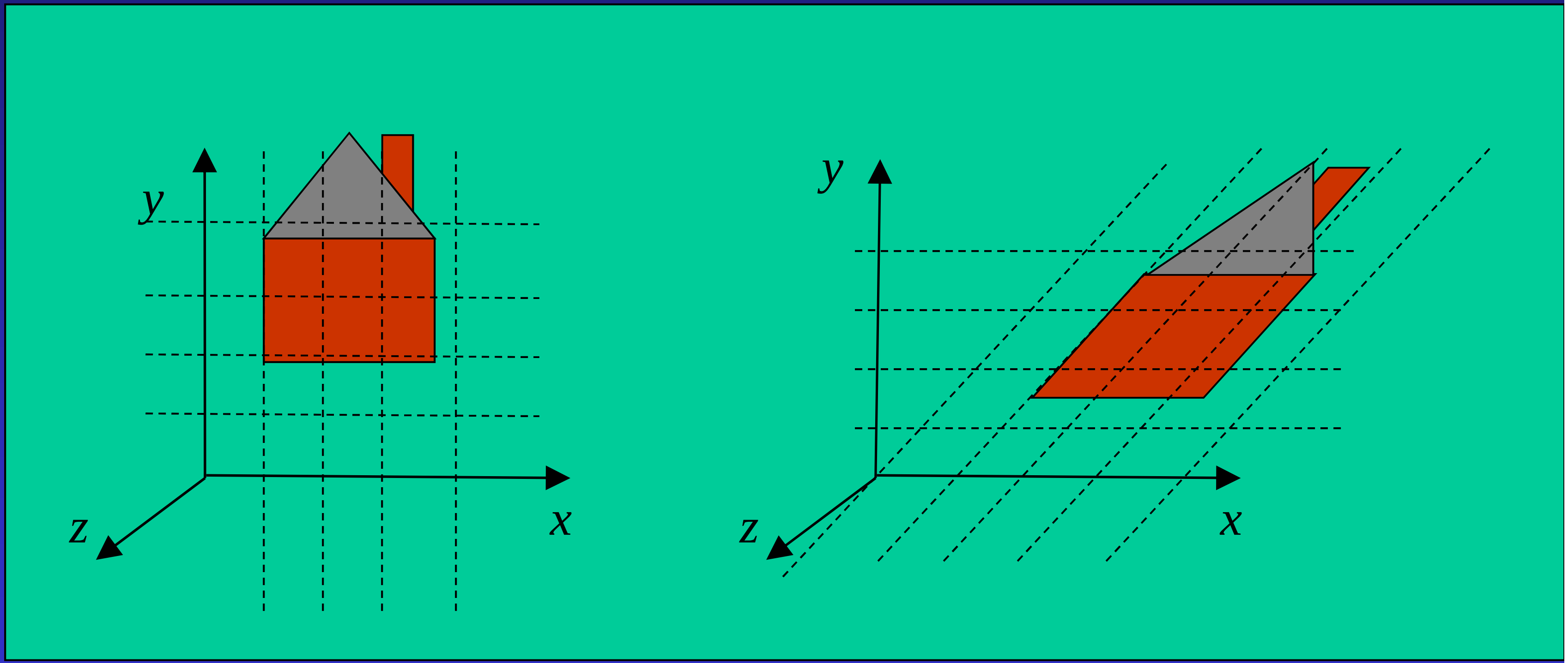
transformed point on a line = point on transformed line

$$M[P(u)]^T = M[P_0 + uV]^T$$

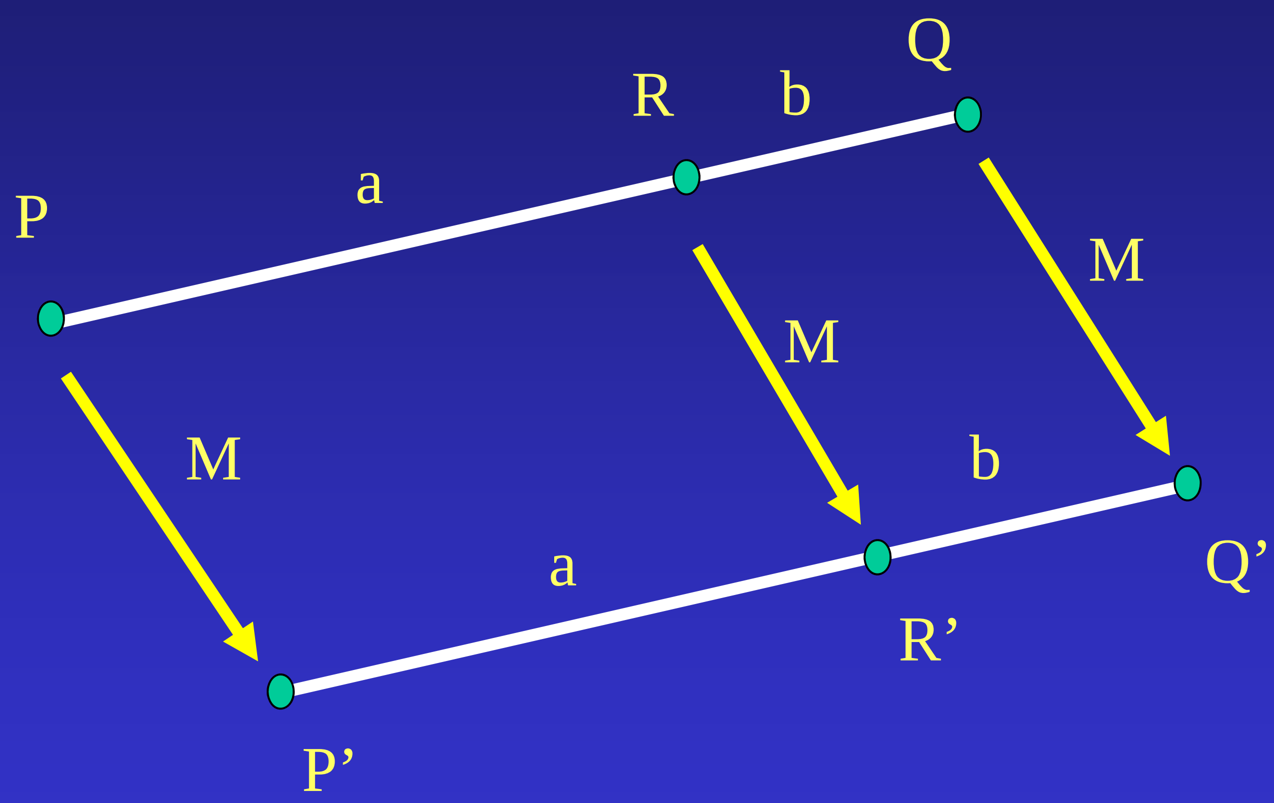
$$M[P_0 + uV]^T = M[P_0]^T + uM[V]^T$$

Theorem: Affine transformations transform lines to lines.

Affine Transformation of Lines



Affine Combinations



Affine Combinations

$$M[R]^T = M[aP + bQ]^T$$

$$M[aP + bQ]^T = M[aP]^T + M[bQ]^T$$

$$M[aP]^T + M[bQ]^T = aM[P]^T + aM[P]^T = aP' + bQ'$$

Theorem: Affine transformations preserve affine combinations.

Properties of Affine Transformations

- Affine transformations map lines to lines;
- Affine transformations preserve affine combinations;
- Affine transformations preserve parallelism;
- Affine transformations change volume by | Det(M) |;
- Any affine transformation can be decomposed into elementary transformations.
- Affine transformations [does/does not] preserve angles?
- Affine transformations [does/does not] preserve the intersection of two lines?
- Affine transformations [does/does not] preserve distances?

Properties of Transformation Matrices

First column is how (1,0,0,0) transforms

$$\begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Second column is how (0,1,0,0) transforms

Third column is how (0,0,1,0) transforms

$$\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix holds how coordinate axes transform and how origin transforms

Properties of Pure Rotation Matrices

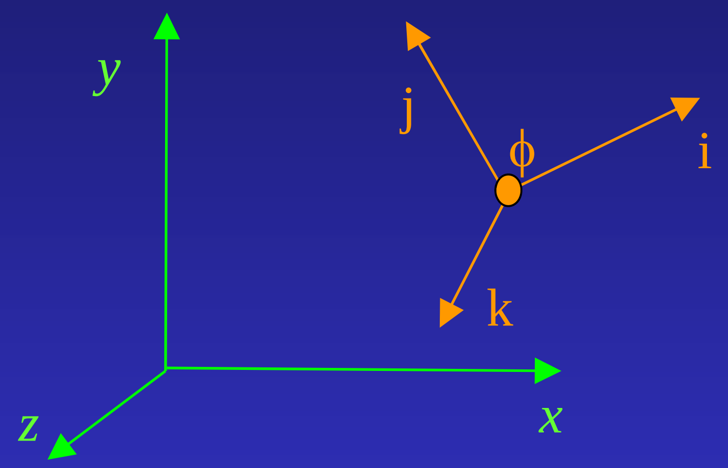
Rows (columns) are orthogonal to each other

A row dot product any other row = 0

Each row (column) dot product times itself = 1

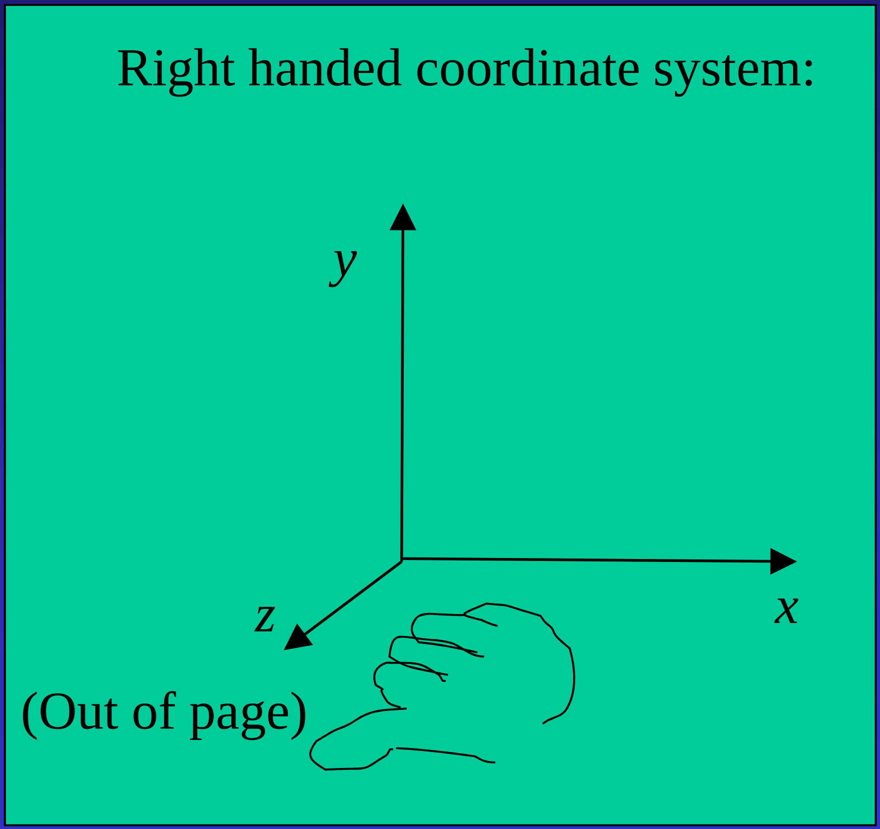
$R^T = R^{-1}$

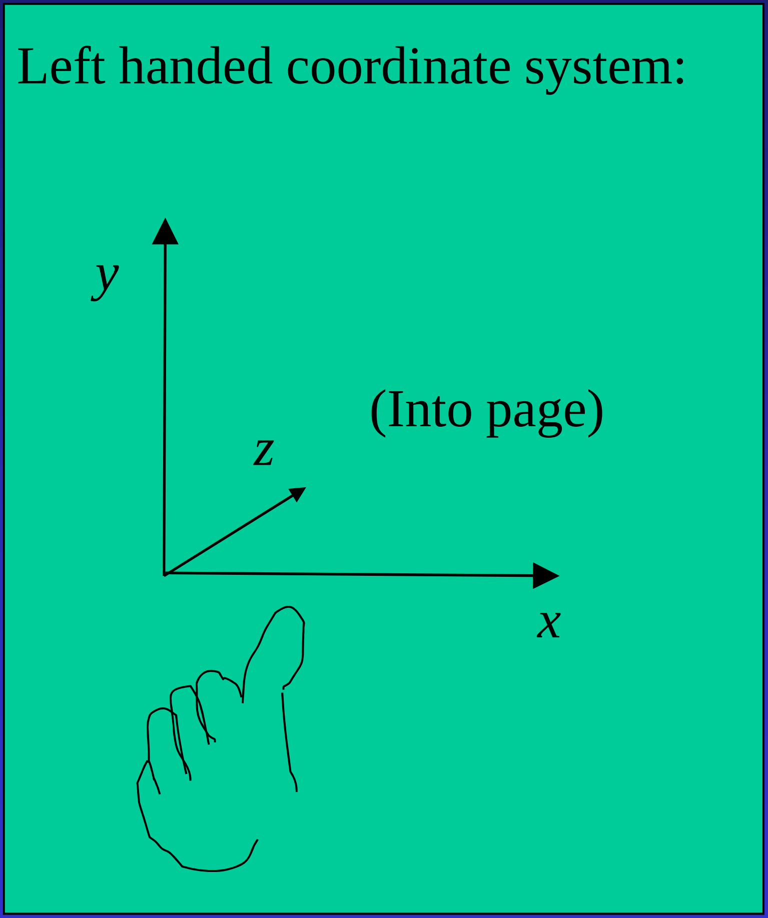
Coordinate Frame



- Coordinate frame is given by origin ϕ and three mutually orthogonal unit vectors, i, j, k - defined in (x,y,z) space
- Mutually orthogonal (dot products): i•j = ?; i•k = ?; j•k = ?.
- Unit vectors (dot products): i•i = ?; j•j = ?; k•k = ?.

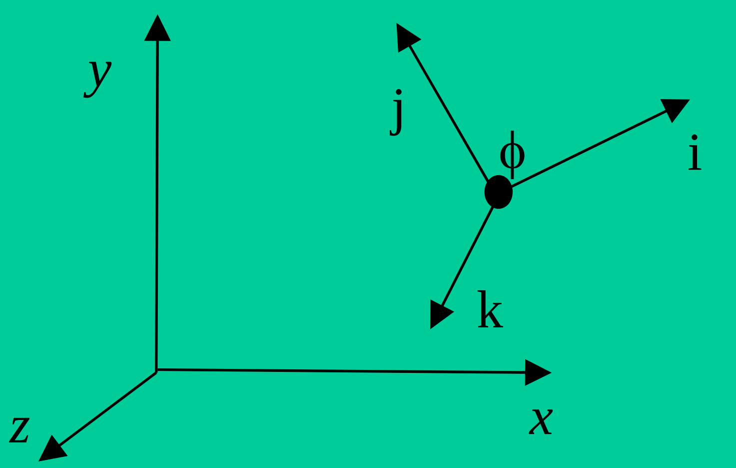
Orientation





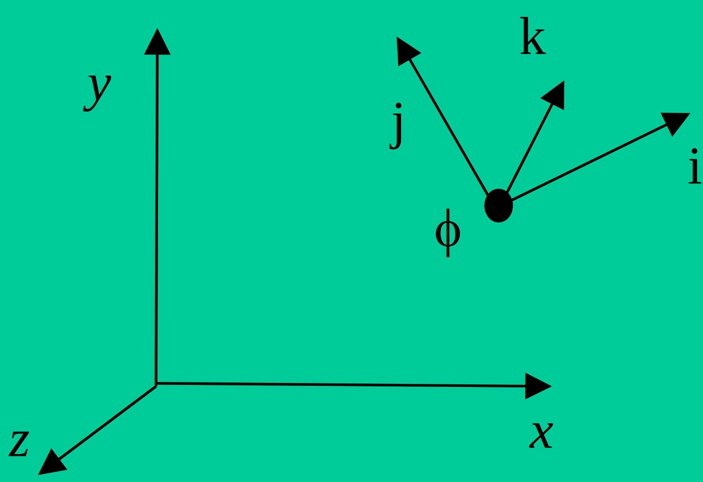
Orientation

Right handed coordinate system:



Cross product: i x j = ?

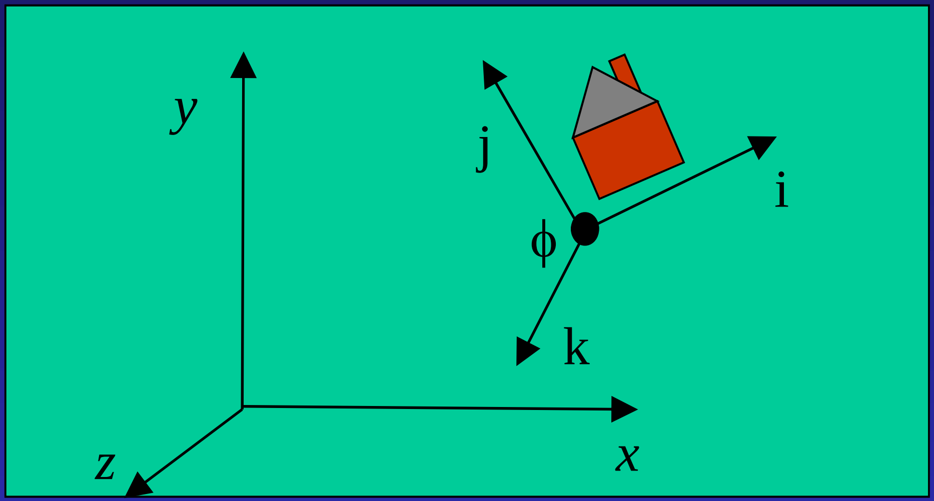
Left handed coordinate system:



Cross product: i x j = ?

How do you test whether (i,j,k) is left handed or right handed?

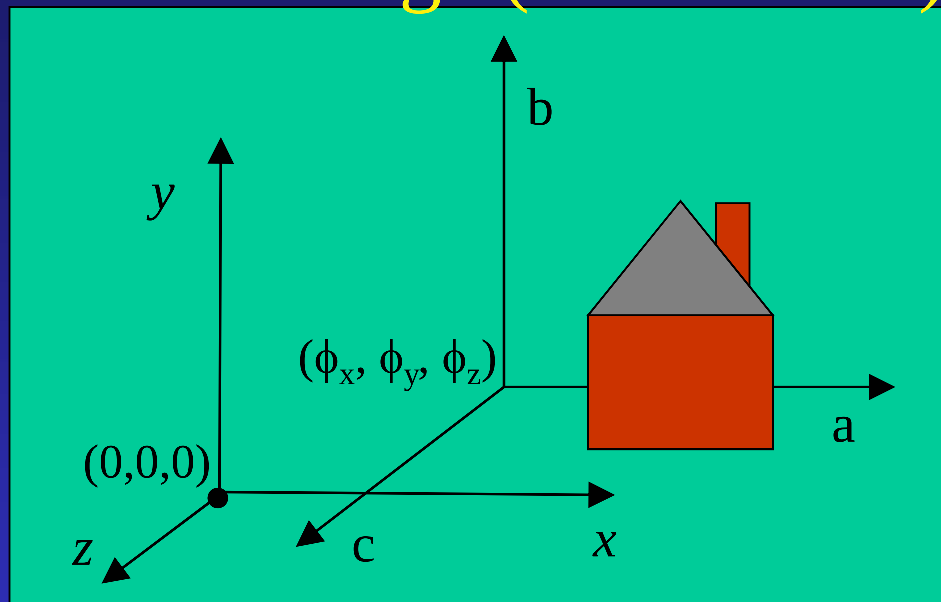
Coordinate Transformations



Given object data points defined in the (i, j, k, ϕ) coordinate frame,
Given the definition of (i, j, k, ϕ) in (x, y, z) coordinates,

How do you determine the coordinates of the object data points in the (x, y, z, 0) frame?

Coordinate change (Translation)

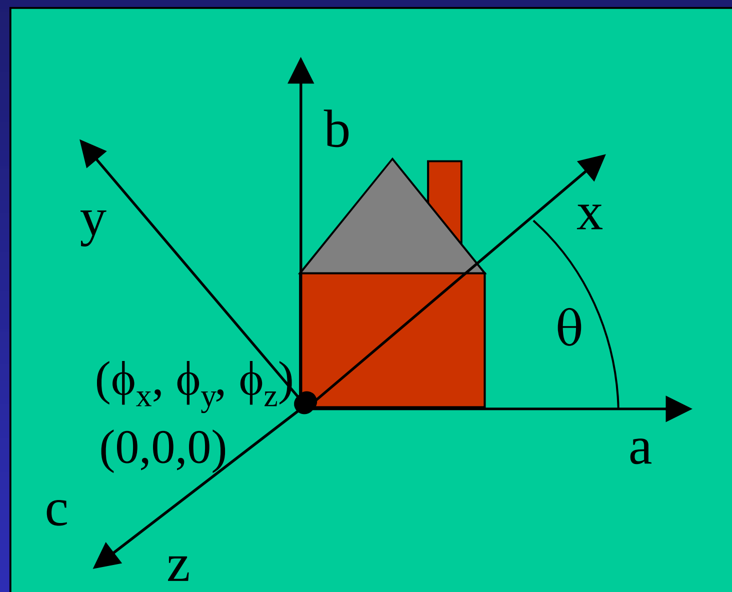


Change from (a,b,c,ϕ) coordinates to (x,y,z,0) coordinates:

1. Move (a,b,c, ϕ) to (x,y,z,0) and invert
2. Move data relative from (0,0,0) out to ϕ position by adding ϕ

$$\begin{pmatrix}x\\y\\z\\1\end{pmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{pmatrix}a\\b\\c\\1\end{pmatrix}.$$

Coordinate change (Rotation)



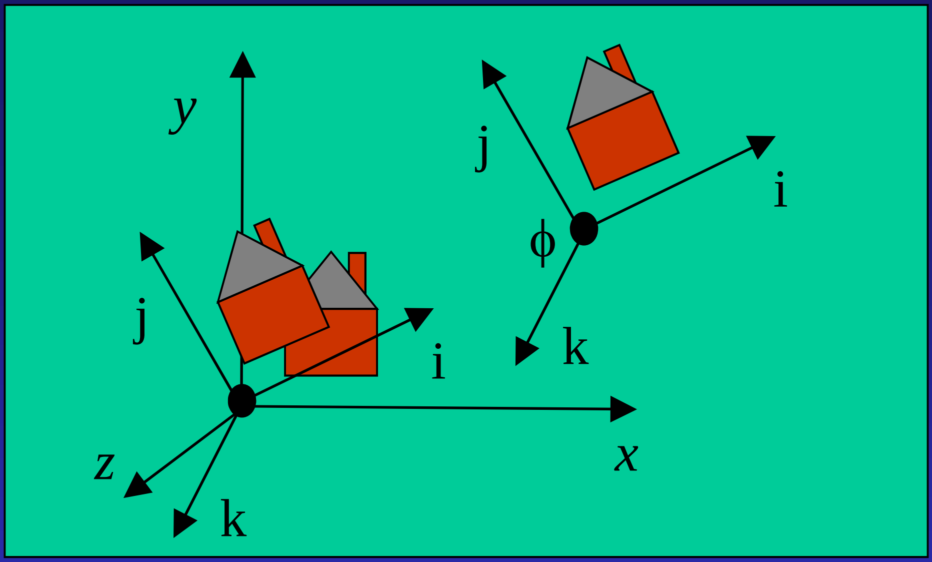
z-axis rotation by θ

Change from (a,b,c,φ) coordinates to (x,y,z,0) coordinates:

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix}.$$

1. Rotate (a,b,c) by θ and invert
2. Rotate data by -θ

Object Transformations

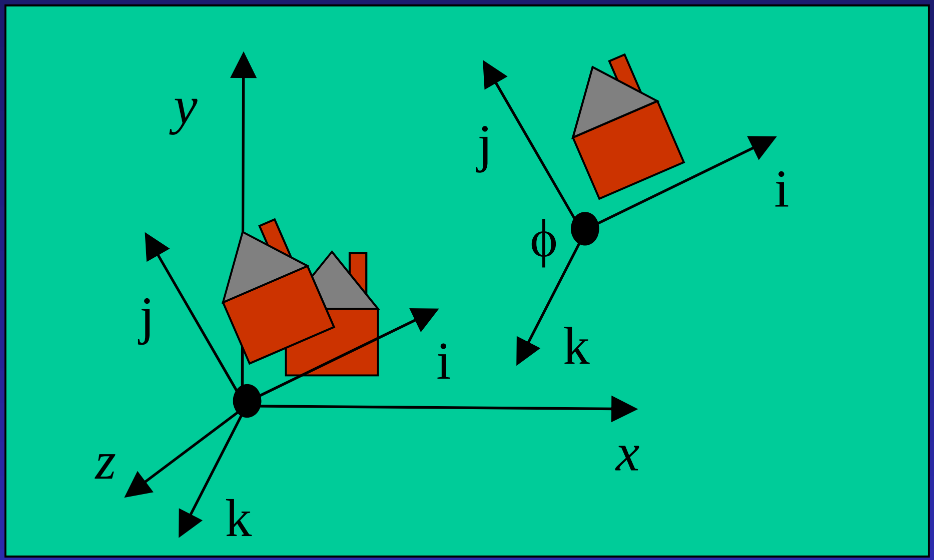


Given (i,j,k,p) defined in the (x,y,z,0) coordinate frame, Transform points defined in the (i, j, k, ϕ) coordinate frame to the (x,y,z,0) coordinate frame.

Rotate object to get x to line up with i, then translate to ϕ

$$\begin{bmatrix} 1 & 0 & 0 & \phi_x \\ 0 & 1 & 0 & \phi_y \\ 0 & 0 & 1 & \phi_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \begin{bmatrix} i_x & j_x & k_x & 0 \\ i_y & j_y & k_y & 0 \\ i_z & j_z & k_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

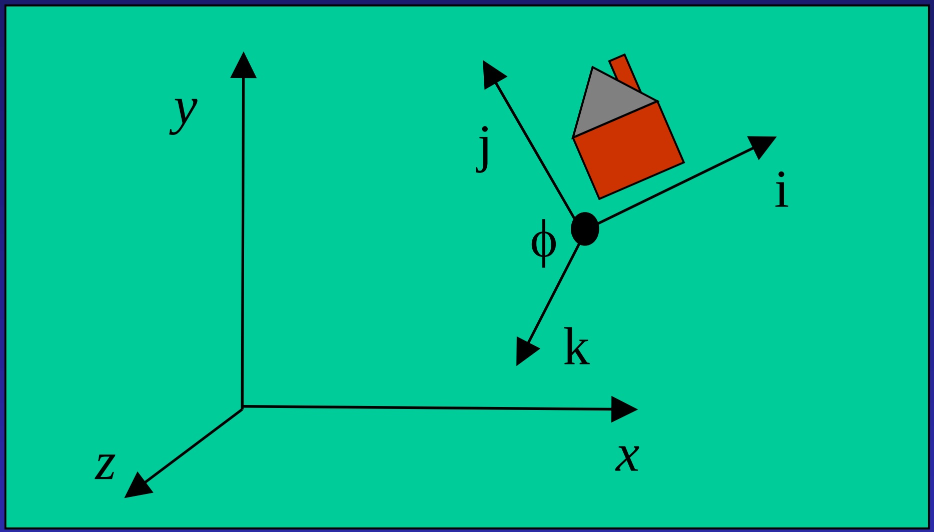
Object Transformations



Affine transformation matrix:

$$\begin{bmatrix} i_x & j_x & k_x & \phi_x \\ i_y & j_y & k_y & \phi_y \\ i_z & j_z & k_z & \phi_z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

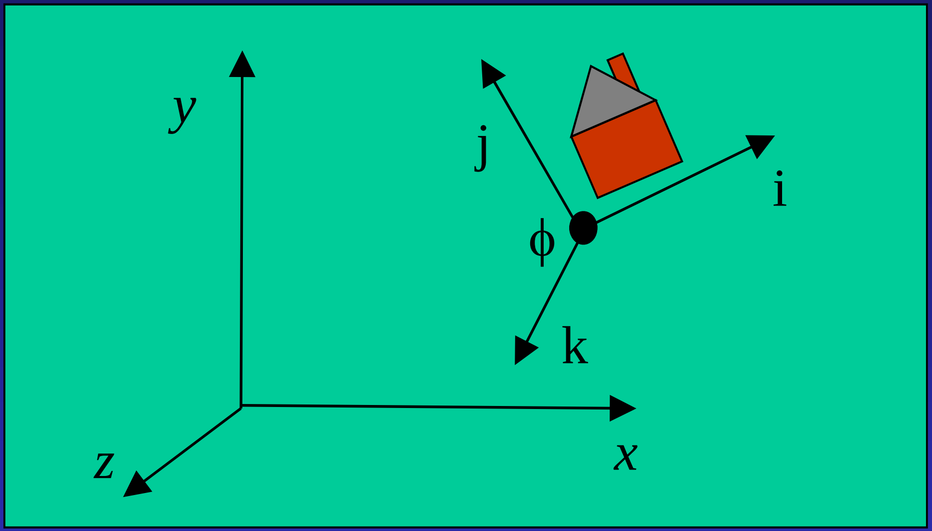
Coordinate Transformations



Given (i,j,k,ϕ) defined in the (x,y,z,0) coordinate frame,
Transform points in (i,j,k,0) coordinate frame to (x,y,z, ϕ) coordinate frame.

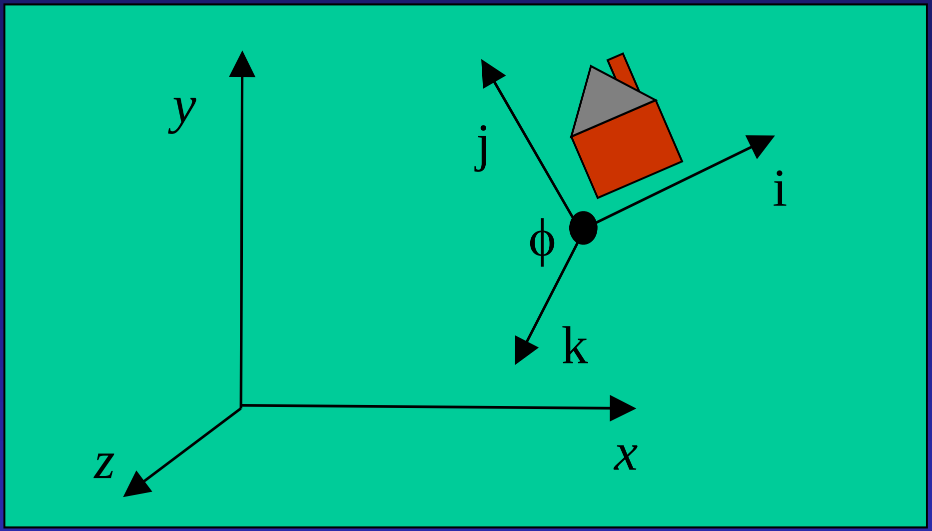
translate by - ϕ, then rotate to align i with x – then invert

Coordinate Transformations



$$R = \begin{bmatrix} i_x & i_y & i_z & 0 \\ j_x & j_y & j_z & 0 \\ k_x & k_y & k_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & -\phi_x \\ 0 & 1 & 0 & -\phi_y \\ 0 & 0 & 1 & -\phi_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Coordinate Transformations



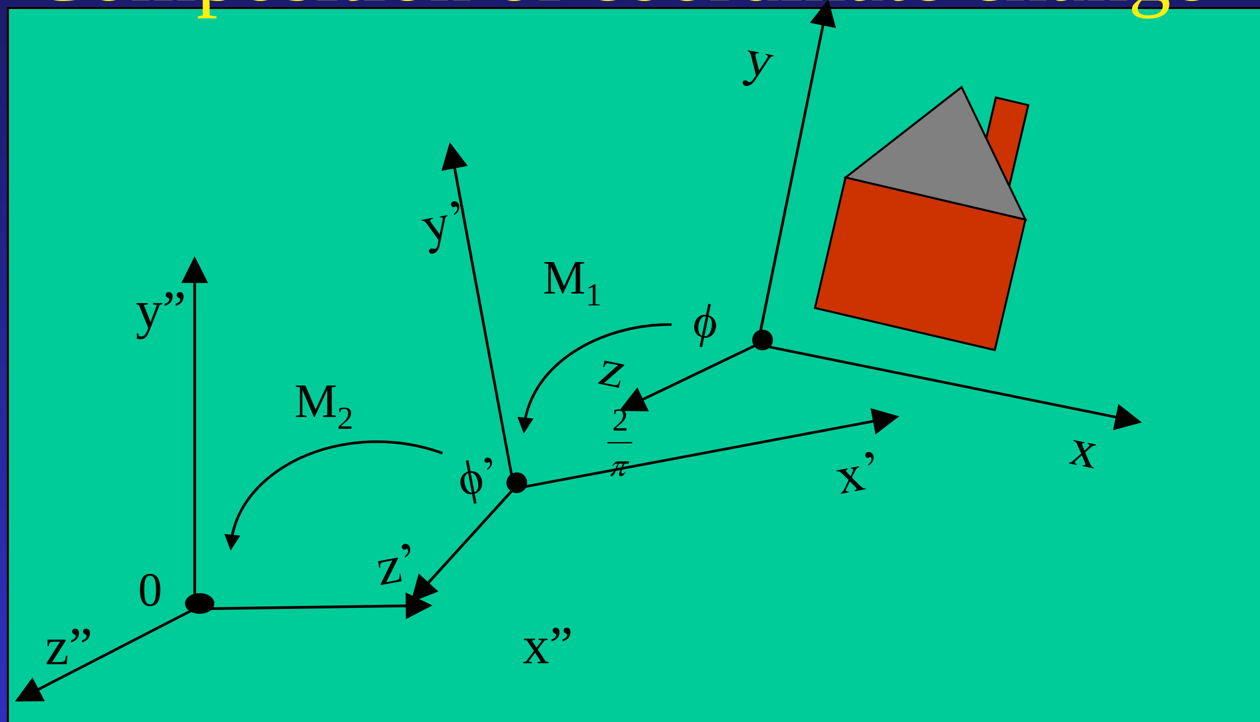
Apply RT to coordinate system

Apply $(RT)^{-1}$ to data = $T^{-1}R^{-1}$

Affine transformation matrix:

$$\begin{bmatrix} i_x & j_x & k_x & \phi_x \\ i_y & j_y & k_y & \phi_y \\ i_z & j_z & k_z & \phi_z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Composition of coordinate change

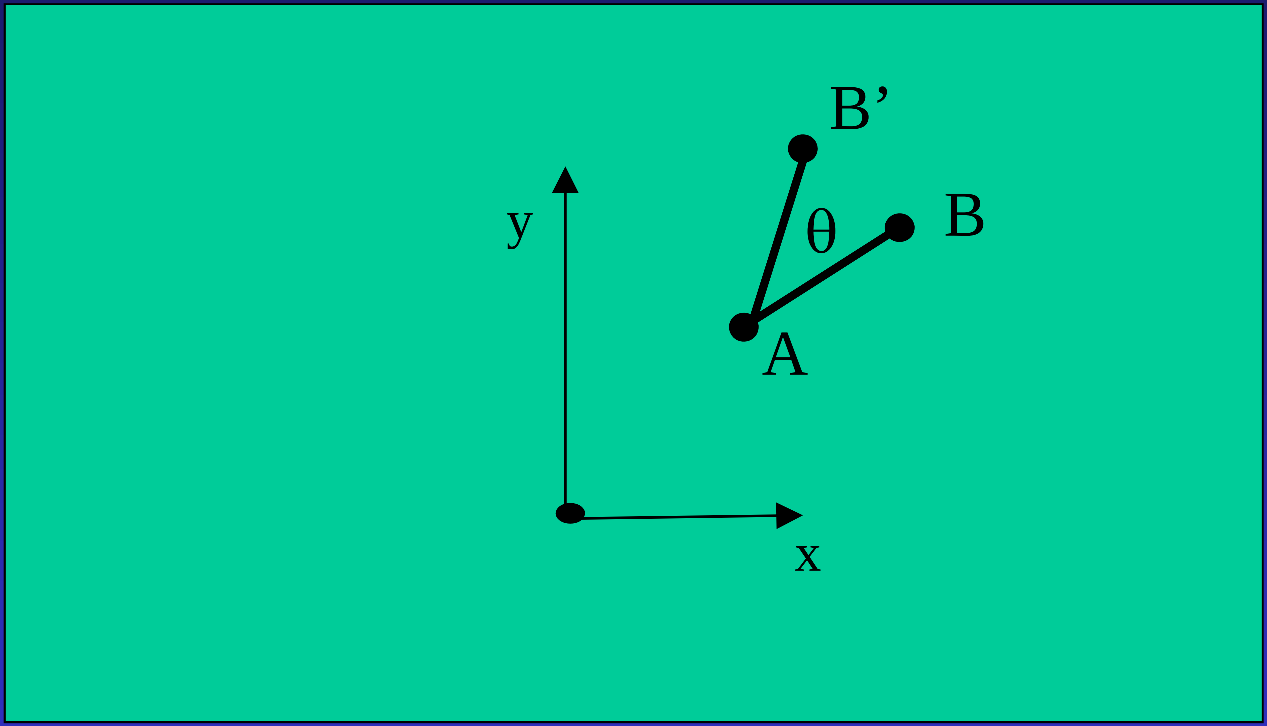


M_1 changes from coordinate frame (x,y,z,θ) to (x’,y’,z’,θ’).

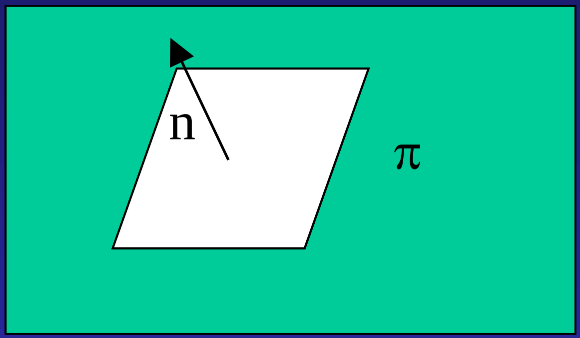
M_2 changes from coordinate frame (x’,y’,z’,θ’) to (x’’,y’’,z’’,θ’’).

Change from coordinate frame (x,y,z,θ) to (x’’,y’’,z’’,0): ?

Composition of transformations - example



Transformations of normal vectors



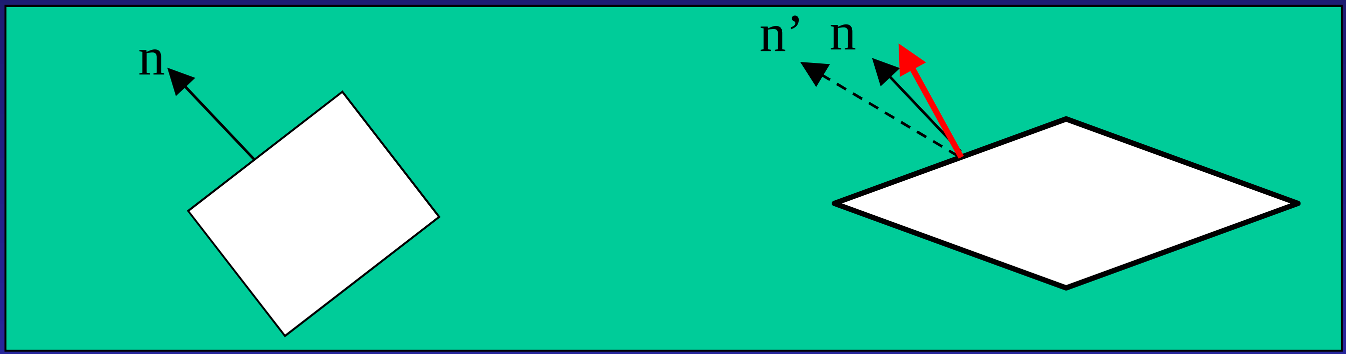
n is a unit normal vector to plane π.

M is an affine transformation matrix.

How is n transformed, to keep it perpendicular to the plane, under:

- translation?
- rotation by θ?
- uniform scaling by s?
- shearing or non-uniform scaling?

Non-uniform scale transformation



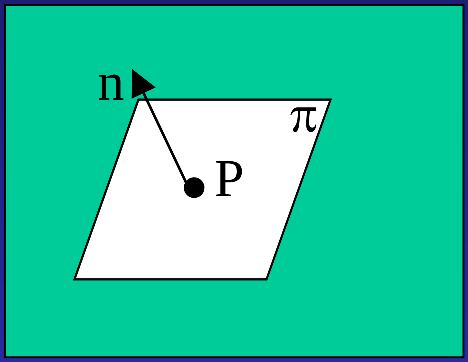
n is a unit normal vector to plane π: $(-n_x,0)$

M is a non-uniform scale transformation matrix that only modifies x-coordinates

If we just transform the n as a vector, then $M\ n^T = n'$

But we want n'' – that has a non-zero y-coordinate value

Transformations of normal vectors



Planar equation: $a\,x + b\,y + c\,z + d = 0$.

Let $N = (a, b, c, d)$. (Note: (a, b, c) is normal vector)

Let $P = (x, y, z, 1)$ be a point in plane π.

Planar equation: $N \cdot P = 0$.

M is an affine transformation matrix. (M^T is M transpose.)

Let $P' = M\,P^T$.

Find N' such that $N' \cdot P' = 0$ (transformed planar equation)

$N \cdot P = 0;$

$N\,P^T = 0;$

$N\,(M^{-1}\,M)\,P^T = 0;$

$(N\,M^{-1})\,(M\,P^T) = 0;$

$(N\,M^{-1})\,P'^T = 0;$

So $N' = NM^{-1}$

To put in column-vector form, $(N')^T = (NM^{-1})^T = (M^{-1})^T N^T$

So $N' = ((M^{-1})^T\,N^T)^T$ and $(M^{-1})^T$ is the transformation matrix to take N^T into N'^T

Note: If M is a rotation matrix, $(M^{-1})^T = M$.