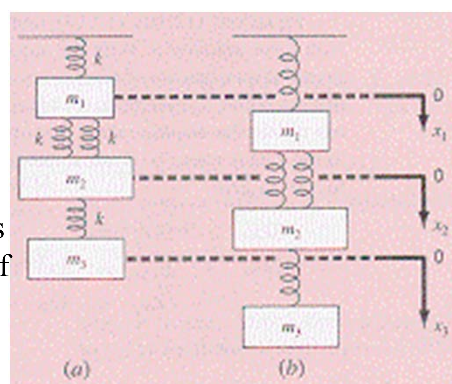
CSE 541 – Numerical Methods

Linear Systems

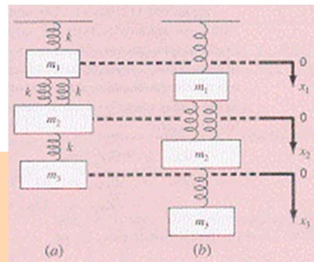
Example

- Suppose we have three masses all connected by springs.
- Each spring has the same constant k.
- Simple force balance gives us accelerations in terms of displacements.



Simple Force Equation

- Recall from elementary physics, that ***F=ma***, or ***ma=F***).



$$m_1 \frac{d^2}{dt^2} x_1 = 2k(x_2 - x_1) + m_1 g - kx_1$$

$$m_2 \frac{d^2}{dt^2} x_2 = k(x_3 - x_2) + m_2 g - 2k(x_2 - x_1)$$

$$m_3 \frac{d^2}{dt^2} x_3 = m_3 g - k(x_3 - x_2)$$

Simple Force Equation

- If we attach the masses and then let go, physically we know that it will oscillate
- Crucial question is what is the steady state
  - i.e., no acceleration

$$\begin{aligned} 3kx_1 - 2kx_2 + 0 &= m_1 g \\ -2kx_1 + 3kx_2 - kx_3 &= m_2 g \\ 0 - kx_2 + kx_3 &= m_3 g \end{aligned}$$

- How do we solve such a linear system of equations?
- Occurs in many circumstances: mass balances, circuit design, stress-strain, weather forecasting, light propagation, etc.

Systems of Equations

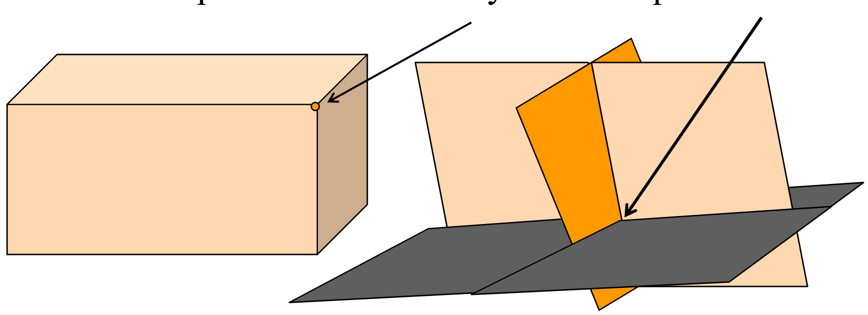
- This simple example produces 3 equations in three unknowns:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \qquad (1)\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \qquad (2)\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \qquad (3)
\end{aligned}
$$

- Geometrically this represents 3 planes in space.

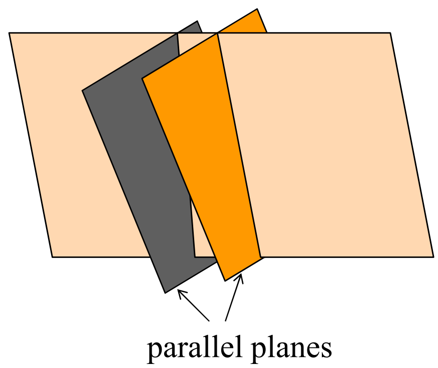
Systems of Equations

- Three different things can happen:
  - Planes intersect at a single point.
  - A unique solution to the system of equations.



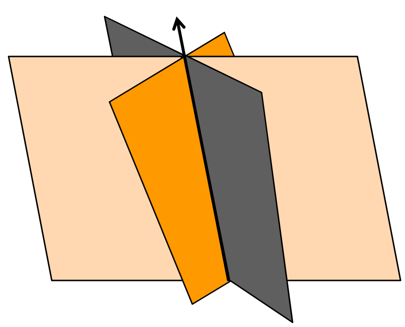
Systems of Equations

- Planes do not intersect at all: (At least two are parallel).



Systems of Equations

- Planes intersect at an infinite number of points (plane or line).



Systems of Equations

- How do we know whether a unique solution exists?
- How do we find such a solution?

Systems of Equations

- In general, we may have *n* equations in *n* unknowns.
- Can we find a solution?
- Can we program an algorithm to efficiently find a solution?
- Is it well behaved? Accuracy? Convergence? Stability?

What is a Matrix?

- A matrix is a set of elements, organized into rows and columns

rows

columns

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A = [a_{ij}]_{mxn} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrix Definitions

- n x m Array of Scalars (n Rows and m Columns)
  - n: row ***dimension*** of a matrix, m: column ***dimension***
  - $m = n$ ➔ ***square matrix*** of dimension n
  - Element $\{a_{ij}\}, \; i = 1, \dots, n, \; j = 1, \dots, m$

$$\mathbf{A} = [a_{ij}]$$

Matrix Definitions

- Column Matrices and Row Matrices
  - ***Column matrix*** (n x 1 matrix):

$$\mathbf{b} = [b_i] = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

  - ***Row matrix*** (1 x n matrix):

$\mathbf{a} = [\mathbf{a_i}] = [a_1\ a_2\ \ldots\ a_n]$

Basic Matrix Operations

- Addition (just add each element)
  - Each matrix must be the same size!

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

- Properties of Matrix-Matrix Addition
  - Commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - Associative: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Basic Matrix Operations

- Subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Basic Matrix Operations

- Scalar-Matrix Multiplication

$$\alpha \mathbf{A} = [\alpha a_{ij}]$$

- Properties of Scalar-Matrix Multiplication

$$\alpha(\beta \mathbf{A}) = (\alpha\beta)\mathbf{A}$$
$$\alpha\beta \mathbf{A} = \beta\alpha \mathbf{A}$$

Basic Matrix Operations

- Matrix-Matrix Multiplication
  - **A**: n x l matrix, **B**: l x m ➔ **C**: n x m matrix

$$\mathbf{C} = \mathbf{AB} = \left[c_{ij}\right]$$

$$c_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj}$$

  - example

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Matrix Multiplication

Matrices A and B have these dimensions:

[n x m] and [p x q]

Matrix Multiplication

Matrices A and B can be multiplied if:

[n x m] and [p x q]

m = p

Matrix Multiplication

The resulting matrix will have the dimensions:

[n x m] and [p x q]

n x q

Computation: A x B = C

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad [2 \times 2]$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \quad [2 \times 3]$$

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

[2 x 3]

Computation: A x B = C

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

[3 x 2] [2 x 3]

A and B can be multiplied

$$C = \begin{bmatrix} 2*1+3*1=5 & 2*1+3*0=2 & 2*1+3*2=8 \\ 1*1+1*1=2 & 1*1+1*0=1 & 1*1+1*2=3 \\ 1*1+0*1=1 & 1*1+0*0=1 & 1*1+0*2=1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 8 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

[3 x 3]

Computation: A x B = C

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

[3 x 2] [2 x 3]

Result is 3 x 3

$$C = \begin{bmatrix} 2*1+3*1=5 & 2*1+3*0=2 & 2*1+3*2=8 \\ 1*1+1*1=2 & 1*1+1*0=1 & 1*1+1*2=3 \\ 1*1+0*1=1 & 1*1+0*0=1 & 1*1+0*2=1 \end{bmatrix} = \begin{bmatrix} 528 \\ 213 \\ 111 \end{bmatrix}$$

[3 x 3]

Matrix Multiplication

- Is AB = BA? Maybe, but maybe not!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \ldots \\ \ldots & \ldots \end{bmatrix} \qquad \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea+fc & \ldots \\ \ldots & \ldots \end{bmatrix}$$

- Heads up: multiplication is NOT commutative!

Matrix Multiplication

- Properties of Matrix-Matrix Multiplication

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

$$\mathbf{AB} \neq \mathbf{BA}$$

The Identity Matrix

- Identity Matrix, **I,** is a Square Matrix:

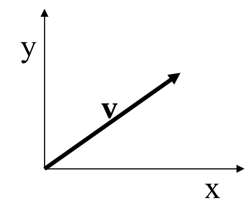
$$\mathbf{I} = [a_{ij}], \quad a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Properties of the Identity matrix:
  - **AI = A   IA = A**
  - Multiplying a matrix with the Identity matrix does not change the initial matrix.

Vector Operations

- Vector: 1 x N matrix
- Interpretation: a line in N dimensional space
- Dot Product, Cross Product, and Magnitude defined on vectors only

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



Matrix Transpose

- ***Transpose***: interchanging the rows and columns of a matrix.

$$\mathbf{A}^T = [a_{ji}]$$

- Properties of the Transpose
  - $(\mathbf{A}^T)^T = \mathbf{A}$
  - $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
  - $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

Inverse of a Matrix

- Some matrices have an inverse, such that:
  $\mathbf{AA^{-1} = I}$, and $\mathbf{A^{-1}A = I}$
- By definition:
  $\mathbf{I^{-1} = I}$, since $\mathbf{I^{-1}I = I^{-1}}$
- Inversion is tricky:
  $\mathbf{(ABC)^{-1} = C^{-1}B^{-1}A^{-1}}$
  Derived from non-commutativity property

Determinant of a Matrix

- Used for inversion
- If det(A) = 0, then A has no inverse
- Can be found using factorials, pivots, and cofactors.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Complexity of Matrix Ops

- Consider a square matrix of $n \times n$ with N elements
- Matrix Addition
  - N additions, so either $O(N)$ or $O(n^2)$
- Scalar-Matrix multiplication
  - N additions, so either $O(N)$ or $O(n^2)$
- Matrix-Matrix multiplication
  - Each element has a row-column dot product.
  - Each element => n multiplications and n-1 additions
  - Total is n^3 multiplications and n^3-n^2 additions, $O(n^3)$

System of Linear Equations

- If our system of equations is ***linear***, then we can write the system as a matrix times a vector of the unknowns equal to the constant terms.

System 1

$x = 3$

$y = 7$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

System 2

$2x + 3y = 2$

$5x - 2y = 24$

$$\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 24 \end{bmatrix}$$

System of Linear Equations

- Examples in three-dimensions

System 3

$$4x - 2y + z = 11$$
$$8x + 5y - 4z = 14$$
$$-3x + y + 5z = 10$$

$$\begin{bmatrix} 4 & -2 & 1 \\ 8 & 5 & -4 \\ -3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 14 \\ 10 \end{bmatrix}$$

System 4

$$4x_1 - 2x_2 + x_3 = 11$$
$$8x_1 + 5x_2 - 4x_3 = 14$$
$$12x_1 + 3x_2 - 3x_3 = 25$$

$$\begin{bmatrix} 4 & -2 & 1 \\ 8 & 5 & -4 \\ 12 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 14 \\ 25 \end{bmatrix}$$

System of Linear Equations

- Each of these examples can be expressed in a simple matrix form:

$$Ax = b$$

- Where **A** is a *nxn* matrix, ***x*** and ***b*** are *nx*1 column matrices (or vectors).

Special Matrices

- Some matrices have special powers or properties:
  - Symmetric matrix
  - Diagonal matrix
  - Lower Triangular matrix
  - Upper Triangular matrix
  - Banded matrix

Symmetric Matrices

- Symmetric matrix – elements are symmetric about the diagonal.

  $\{a_{ij}\} = \{a_{ji}\}$ for all *i,j*

  $a_{12} = a_{21}$, $a_{33}=a_{33}$, etc.

- Implies **A** is equal to its transpose.

  $\mathbf{A} = \mathbf{A}^T$

Diagonal Matrices

- A **diagonal** matrix has zero's everywhere except possibly along the diagonal.

  $\{a_{ij}\} = 0$ for all $i \neq j$.

$$D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

- Addition, scalar-matrix multiplication and matrix-matrix multiplication among diagonal matrices preserves diagonal matrices.

  C = AB $\{c_{ij}\} = 0\ i \neq j;\ \{c_{ii}\} = \{a_{ii}b_{ii}\}$ ← **O**(n)
- All operations are only O(n).

Lower Triangular Matrix

- A ***lower-triangular*** matrix has a value of zero for all elements above the diagonal.

  $\{l_{ij}\} = 0\ i < j.$

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ l_{n1} & \cdots & l_{nn-1} & l_{nn} \end{bmatrix}$$

- Can you solve the ***first*** equation?

Upper-Triangular Matrix

- A ***upper-triangular*** matrix has a value of zero for all elements below the diagonal.

  $\{u_{ij}\} = 0 \ i > j.$

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & & \vdots \\ \vdots & & \ddots & u_{n-1n} \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

- Can you solve the ***last*** equation?

Banded Matrices

- A banded matrix has zeros as we move away from the diagonal.

  $\{b_{ij}\} = 0 \ i > j+b$ and $i < j-b$.

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1b} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ b_{b1} & & \ddots & \ddots & \ddots & & b_{bn} \\ 0 & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & b_{n-1n} \\ 0 & \cdots & 0 & b_{nb} & \cdots & b_{nn-1} & b_{nn} \end{bmatrix}$$

band-width b