CSE 541

ELEMENTARY NUMERICAL METHODS

Functions

Review

function representation

explicit function
implicit function
parametric

types

power series
polynomial

complexity

linear v. non-linear
multi-dimensional

y=f(x)
f(x,y) = 0
x=f(t), y=g(t)

approximate by linear equation, take a step - repeat
e.g., inverse kinematics

power series an*(x-c)^n
polynomial ai x^n y^m; degree of polynomial, of term
transcendental - transcends algebra - finite add, sub, mult, div, root taking, rational coeff
analytic -

Polynomial evaluation - Horner's Method

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

$$f(x) = a_0 + x(a_1 + a_2 x + a_3 x^2 ... + a_n x^{n-1})$$

$$f(x) = a_0 + x(a_1 + x(a_2 + a_3 x ... + a_n x^{n-2})$$

$$f(x) = a_0 + x(a_1 + x(a_2 + x(a_3 + ... + x a_n)...)))))$$

example of optimization (low-level)
higher level: algorithm selection
midlevel: pre-compute values, use table look-ups, interpolate
low level: reorganize instructions

Terminology

Absolute error

Relative error

Rounding

Chopping

Precision

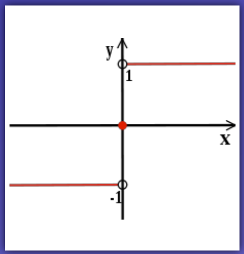
Accuracy

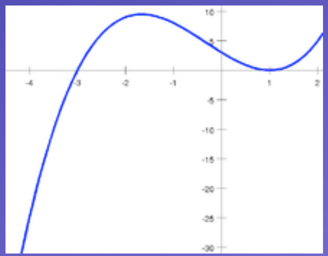
Significant digits

Continuity

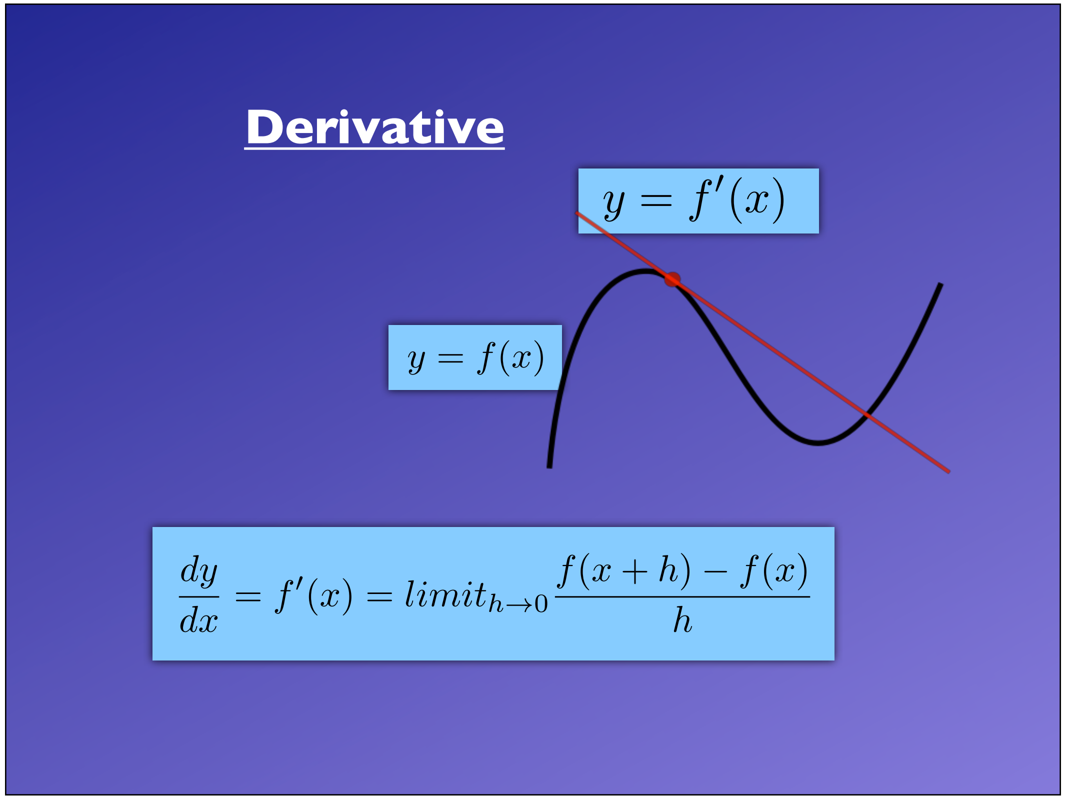
y = f(x)

Small change in input -> small change in output

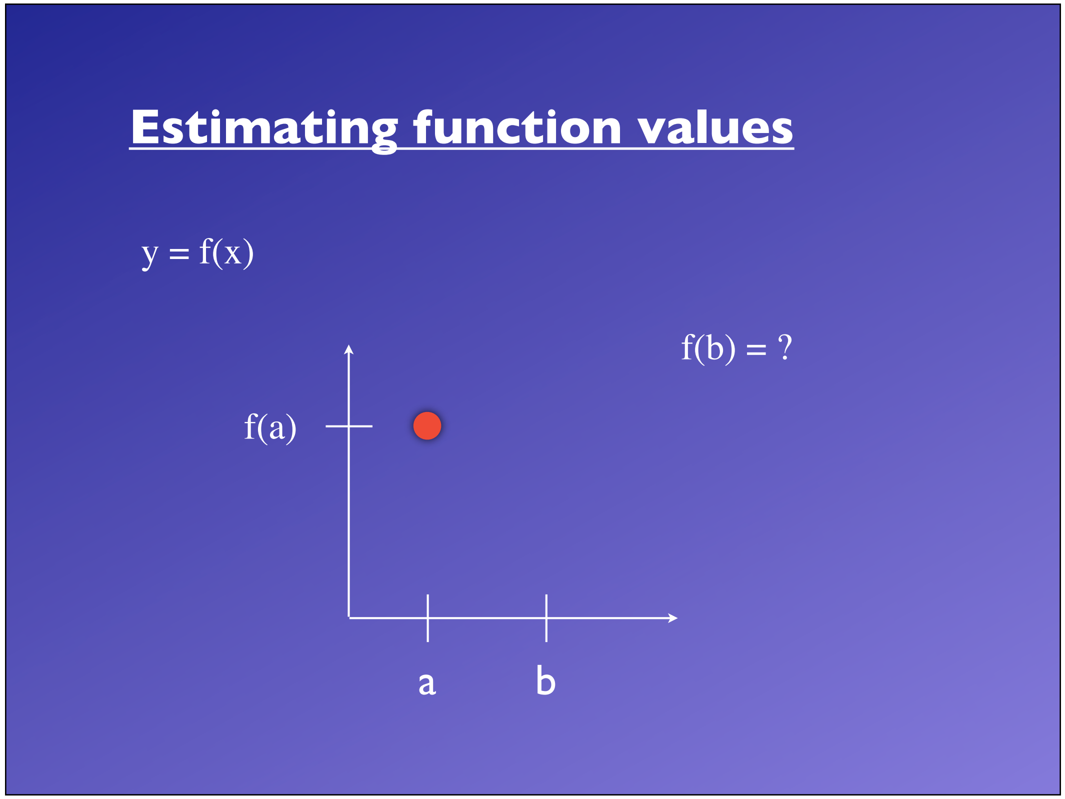




curve that represents f(x)
is unbroken with no gaps



review
instantaneous change in position
tangent, slope
(x,y)’ = (x,f(x))’ = (1,f’(x))
2nd derivative - change in first derivative
curvature

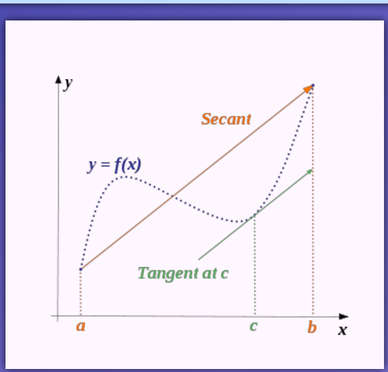


unknown function f(x)
know f(a)
what is f(b)?
error if |f'(x)| a<x<b is less than k

Mean Value Theorem

$$\frac{f(b)-f(a)}{b-a} = f'(\xi) \quad \text{for some } a < \xi < b$$

$$f(b) = f(a) + (b-a)f'(\xi)$$



first series - no significant digits
second series: (4/3) / (2/3) = 2; 4 significant digits

Taylor Series

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(x)}{2!}(x-c)^2 + \frac{f'''(x)}{3!}(x-c)^3 + \ldots$$

$$f(x) \approx \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Maclaurin Series

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$f(x) \approx \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

Taylor series of f at point c
Maclaurin Series is Taylor series at 0

Taylor's Theorem for f(x)

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}$$

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

$$where\ \xi = \xi(x) \in (c, x)$$

why can we ignore higher order terms

Taylor's Theorem for f(x)

$$f(x) \approx \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

what happens at higher derivatives?

|x-c| << 1

|x-c| >> 1

(x-c)^{n+1}
might go to zero or might go to infinity

Common Derivatives

$$\frac{d}{dx}(ax^n) = nax^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\left.\begin{aligned}
&\frac{d}{dx}(au^n) = nau^{n-1}\frac{du}{dx} \\
&\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx} \\
&\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx} \\
&\frac{d}{dx}(e^u) = e^u\frac{du}{dx} \\
&\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}
\end{aligned}\right\} chain\,rule$$

(x-c)^{n+1}
might go to zero or might go to infinity

Example Series

at c = 0

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$

$$= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots \qquad -1 < x < 1$$

(x-c)^{n+1}

might go to zero or might go to infinity

Example Series

at c = 0

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \qquad -1 < x \le 1$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \qquad -1 < x < 1$$

(x-c)^{n+1}
might go to zero or might go to infinity

Example from book

$$\ln 2 = 0.693147180\ldots$$

$$\ln(1+1) \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}$$

$$= 0.63452$$

$$\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) \approx 2\left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \frac{\left(\frac{1}{3}\right)^7}{7} + \cdots\right)$$

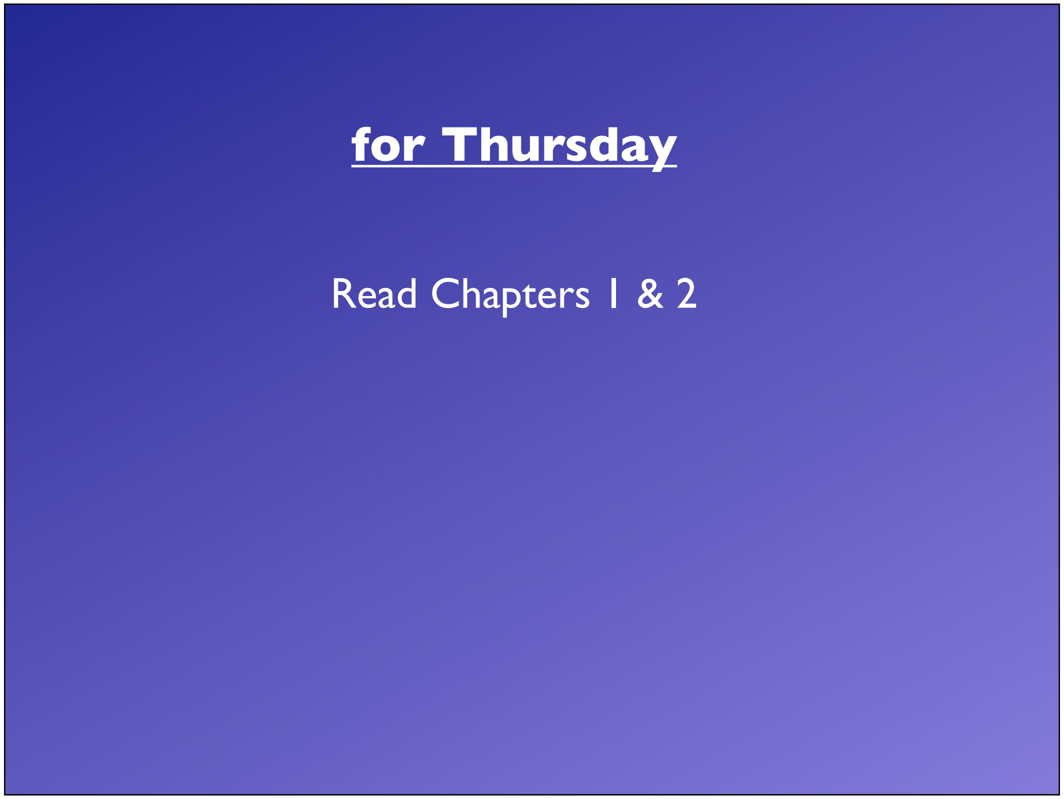
$$= 0.69313$$

first series with 8 terms- no significant digits
second series with 4 terms: (4/3) / (2/3) = 2; 4 significant digits

Alternating Series

If the magnitudes of the terms in an alternating series converge monotonically to zero, then the error in truncating the series is no larger than the magnitude of the first omitted term

first series – no significant digits
second series: (4/3) / (2/3) = 2; 4 significant digits



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