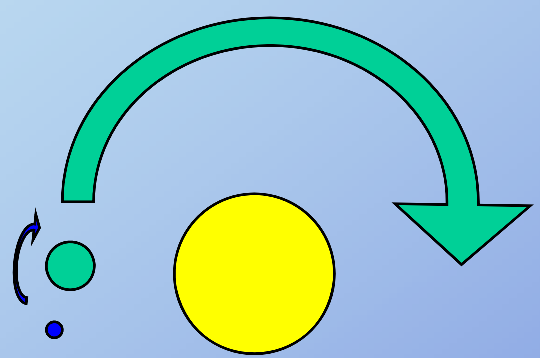
Computer Animation Algorithms and Techniques

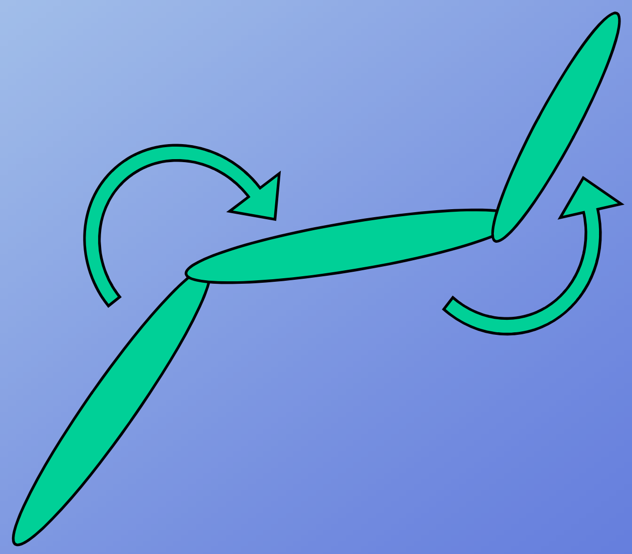
Kinematic Linkages

Hierarchical Modeling

Relative motion → Parent-child relationship

Simplifies motion specification





Constrains motion → Reduces dimensionality

Modeling & animating hierarchies

3 aspects

1. **Linkages & Joints – the relationships**
2. **Data structure – how to represent such a hierarchy**
3. **Converting local coordinate frames into global space**

Some terms

Joint – allowed relative motion & parameters
Joint Limits – limit on valid joint angle values
Link – object involved in relative motion
Linkage – entire joint-link hierarchy
Armature – same as linkage
End effector – most distant link in linkage
Articulation variable – parameter of motion associated with joint
Pose – configuration of linkage using given set of joint angles
Pose vector – complete set of joint angles for linkage

Arc – of a tree data structure – corresponds to a joint
Node – of a tree data structure – corresponds to a link

Use of hierarchies in animation

Forward Kinematics (FK)

animator specifies values of articulation variables

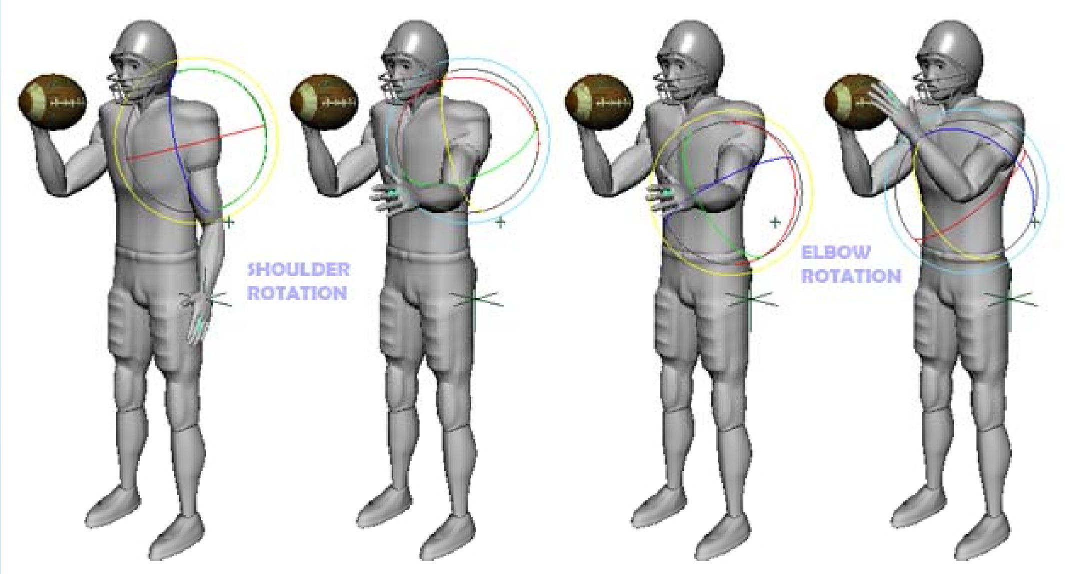
global transform for each linkage is computed

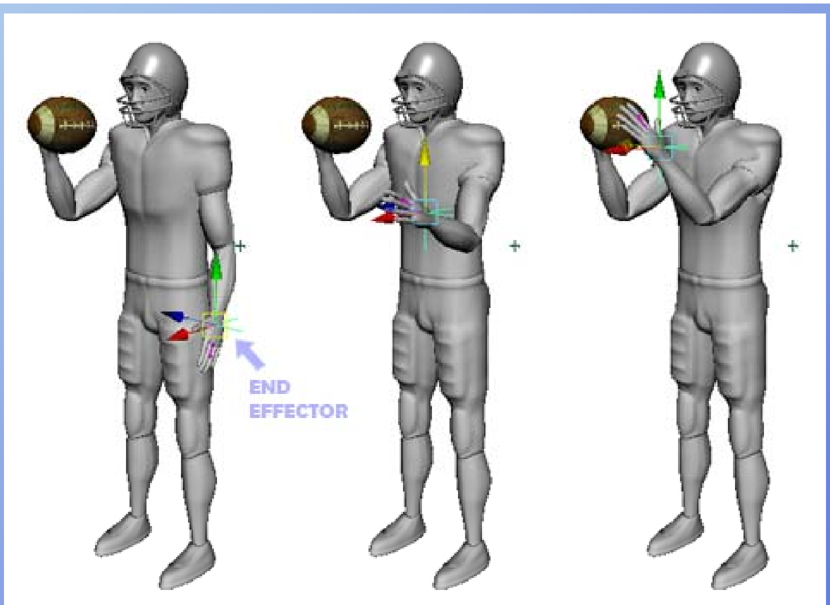
Inverse Kinematics (IK)

animator specifies final desired global transform for end effector (and possibly other linkages)

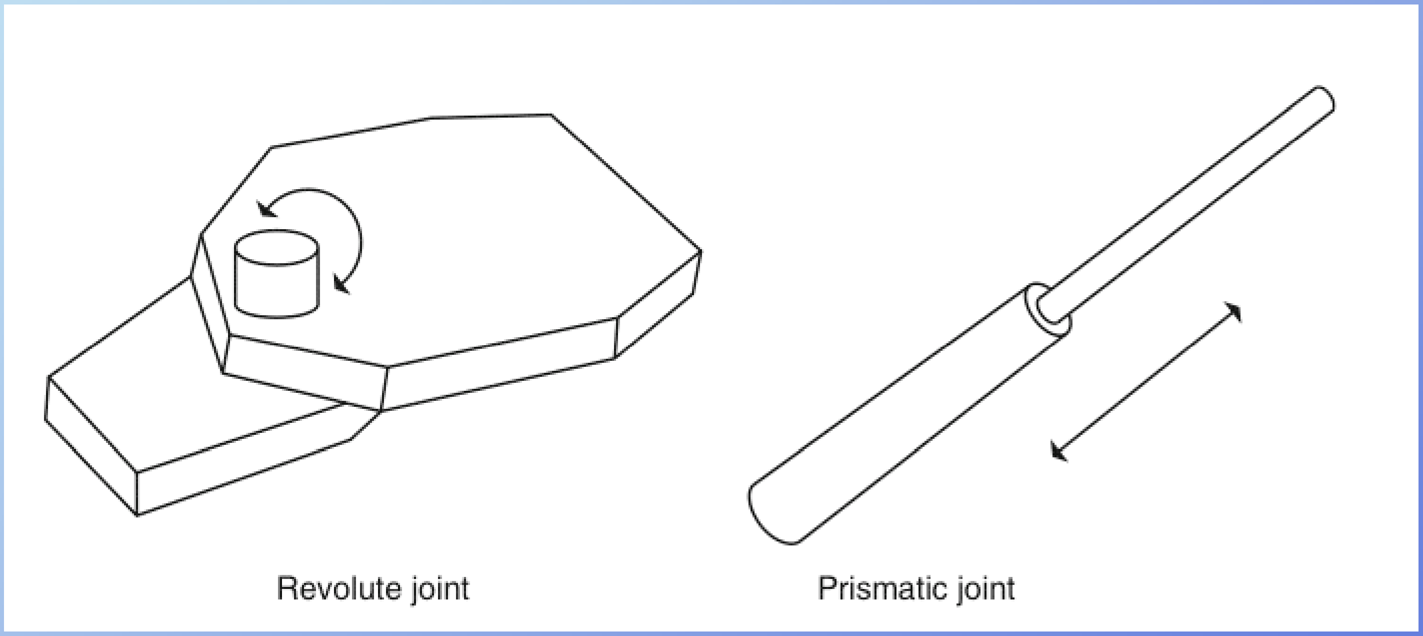
Values of articulation variables are computed

Forward & Inverse Kinematics

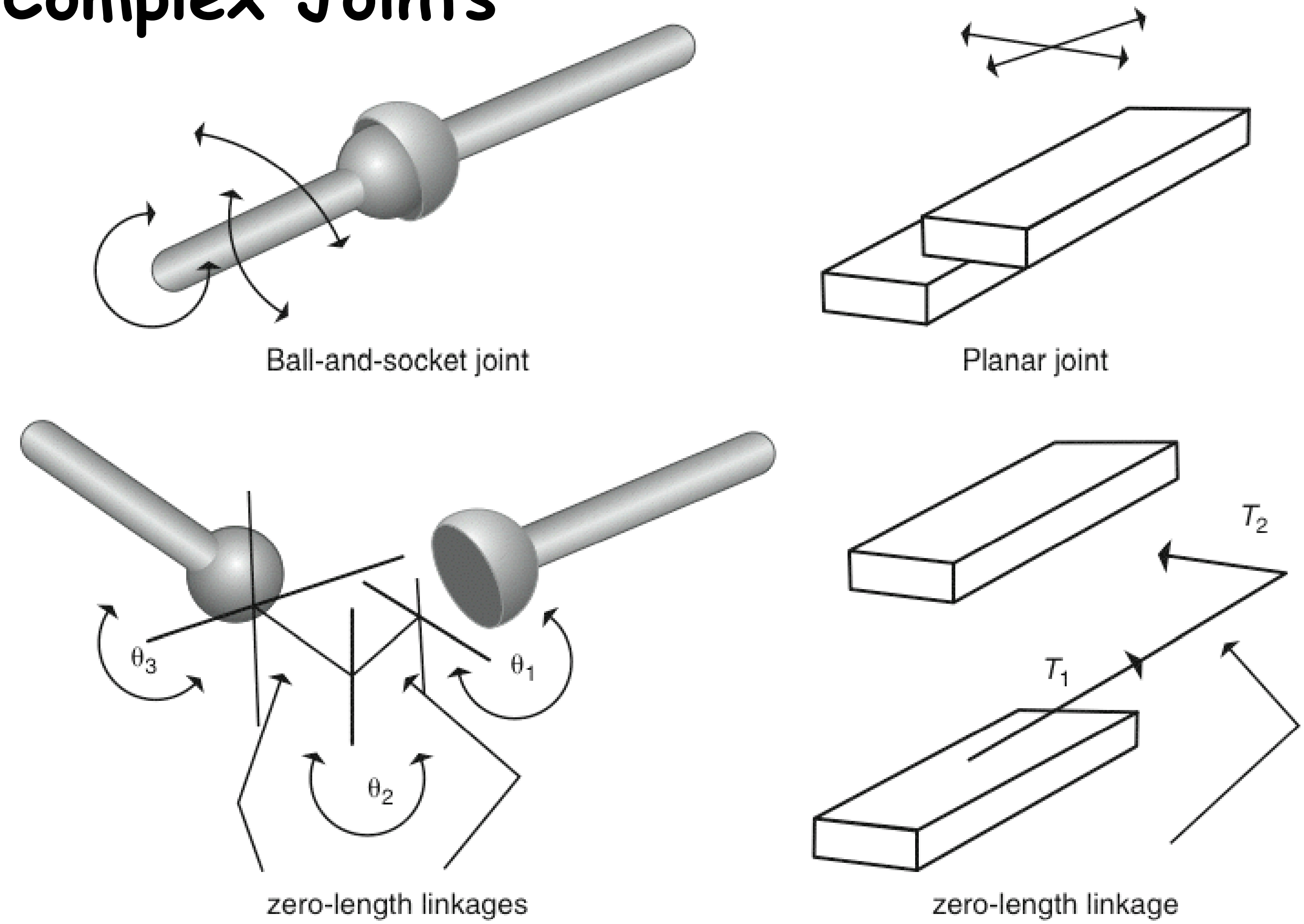




Joints – relative movement



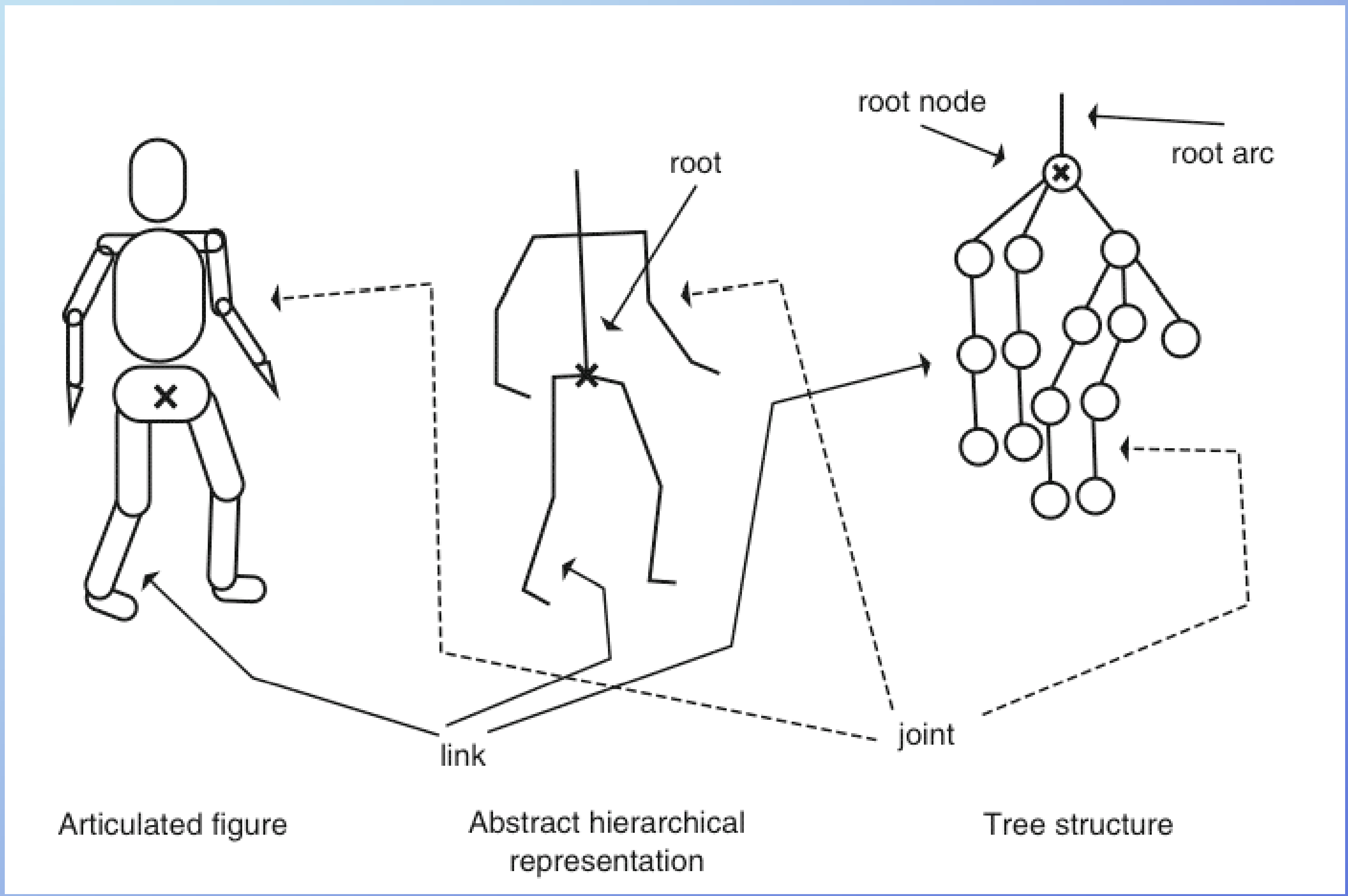
Complex Joints



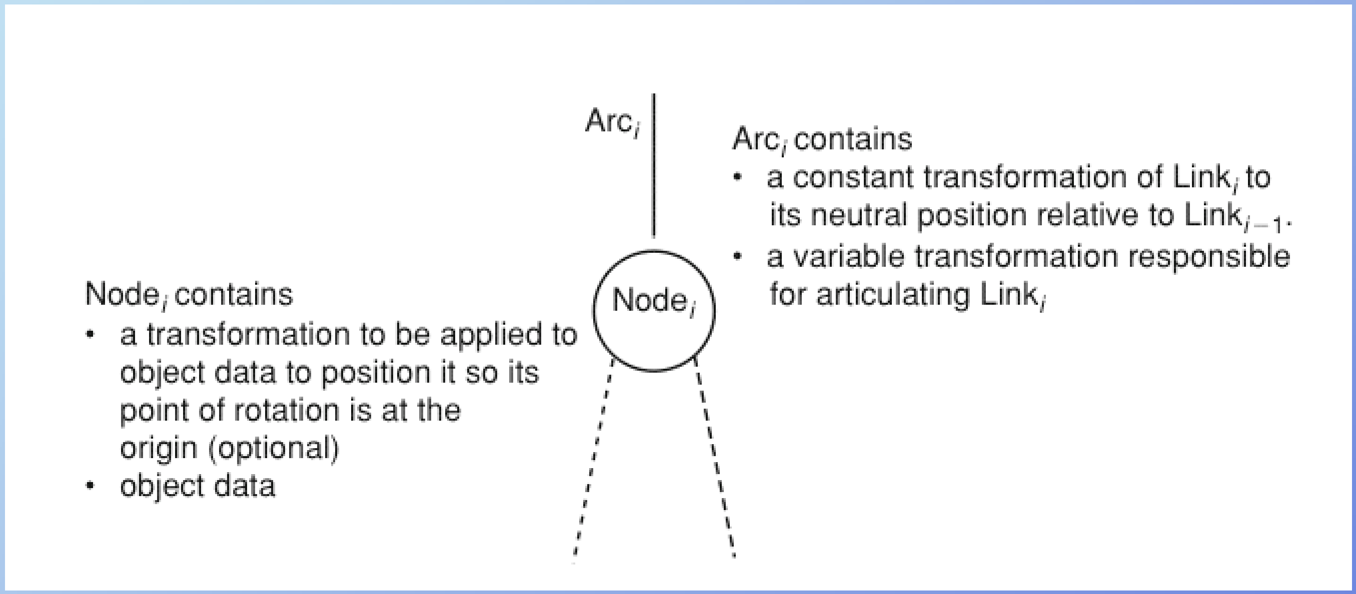
Ball-and-socket joint modeled as 3 one-degree joints with zero-length links

Planar joint modeled as 2 one-degree prismatic joints with zero-length links

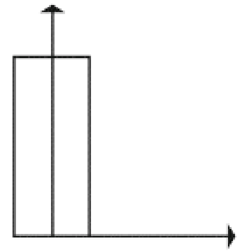
Hierarchical structure



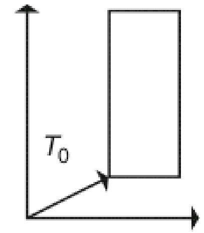
Tree structure



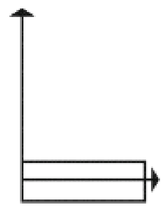
Tree structure



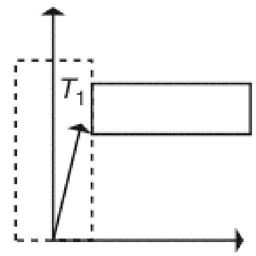
Original definition of root object ($Link_0$)



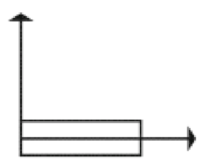
Root object ($Link_0$) transformed (translated and scaled) by T_0 to some known location in global space



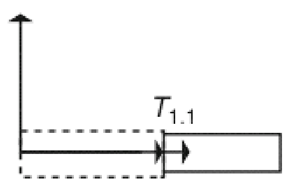
Original definition of $Link_1$



$Link_1$ transformed by T_1 to its position relative to untransformed $Link_0$

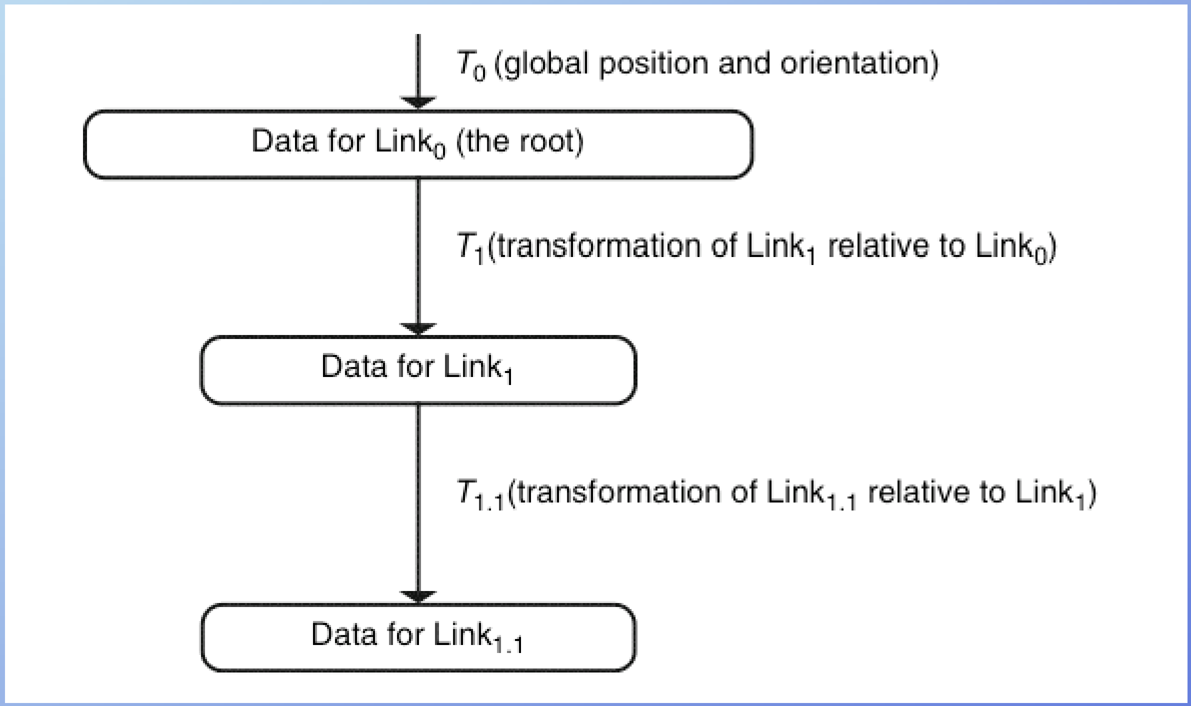


Original definition of $Link_{1.1}$

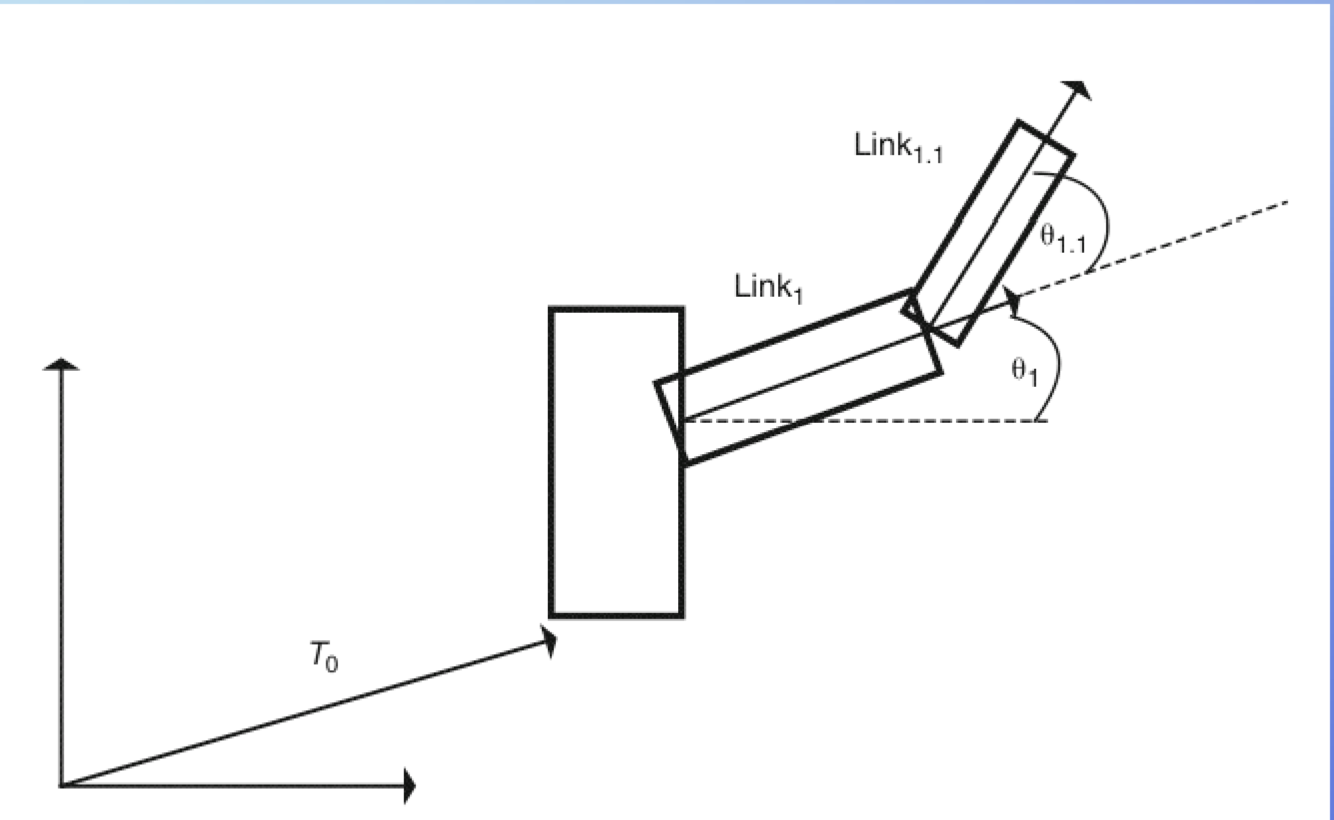


$Link_{1.1}$ transformed by $T_{1.1}$ to its position relative to untransformed $Link_1$

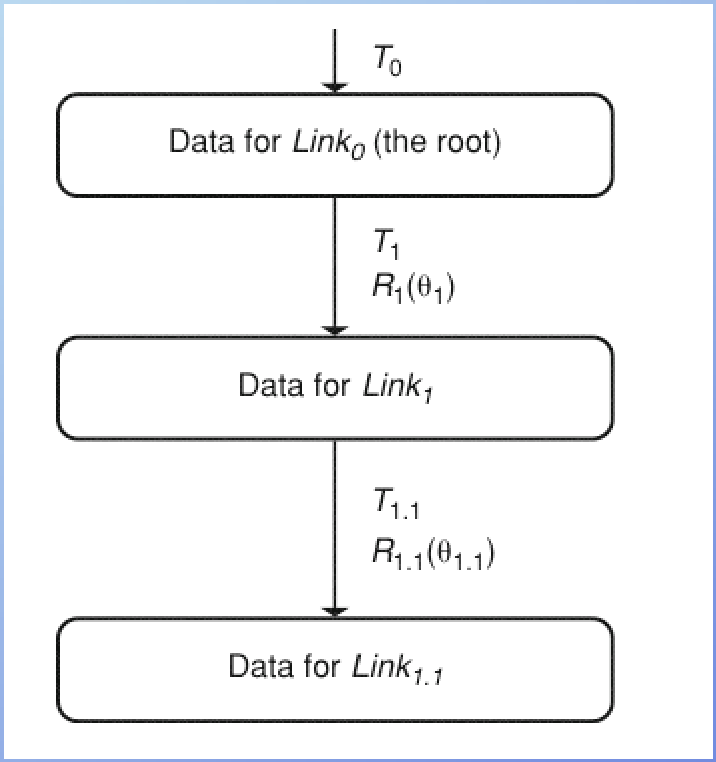
Tree structure



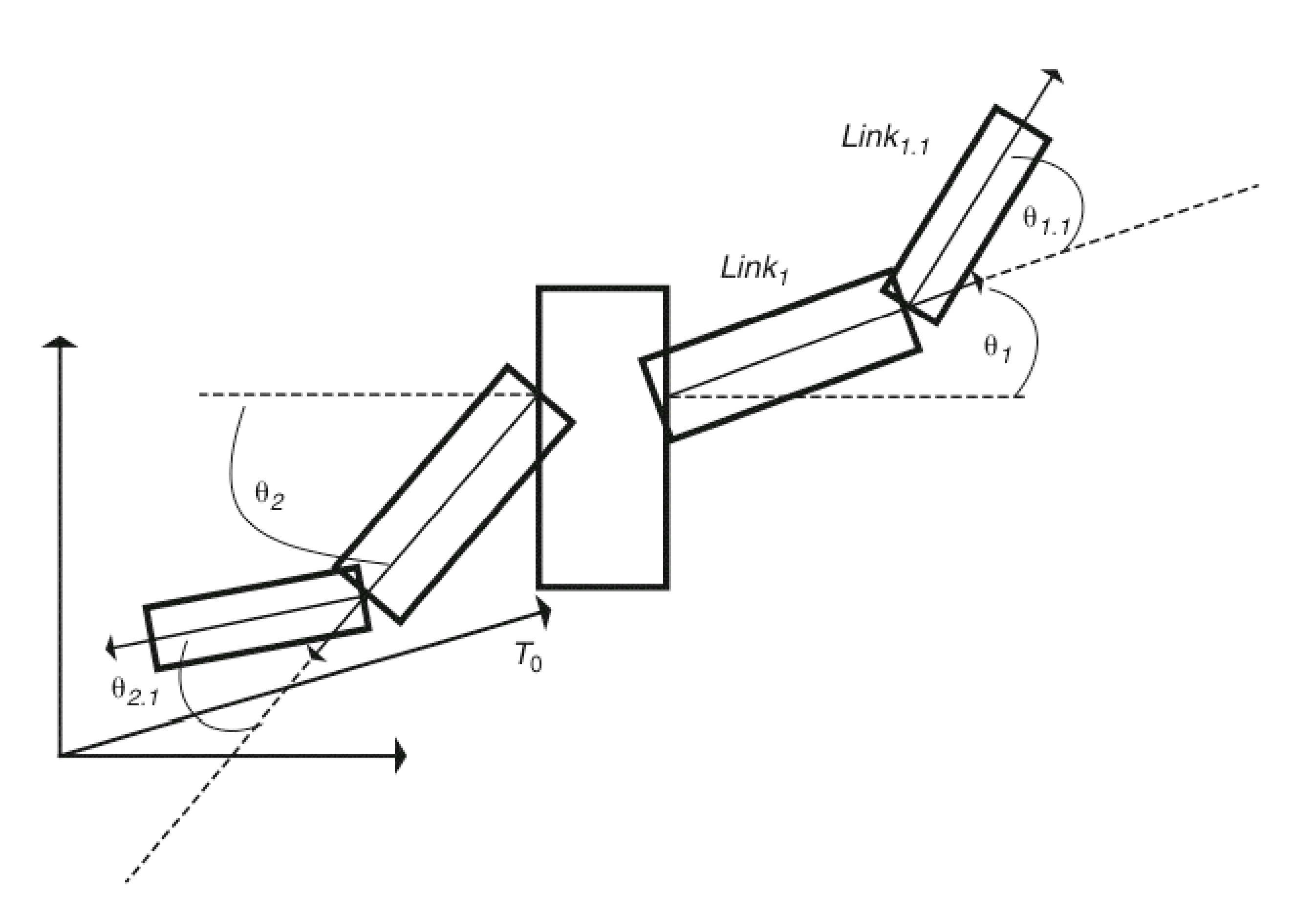
Relative movement



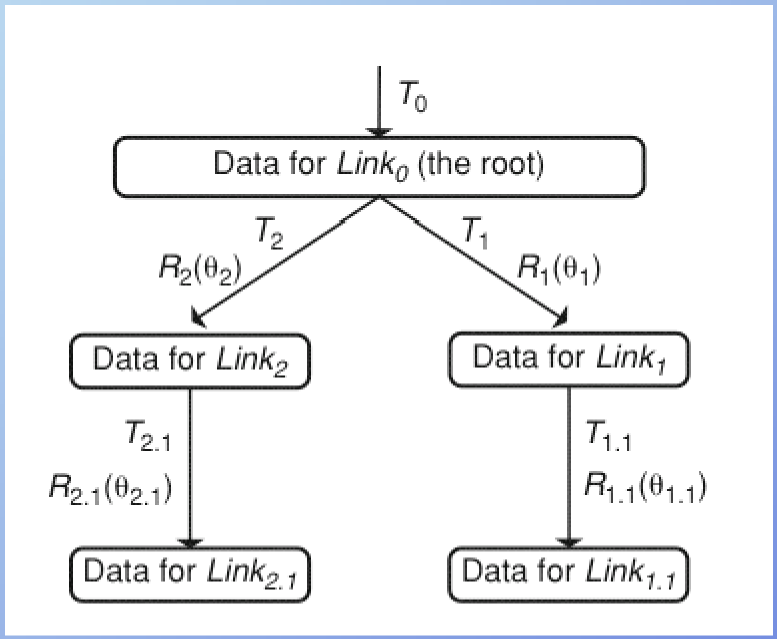
Relative movement



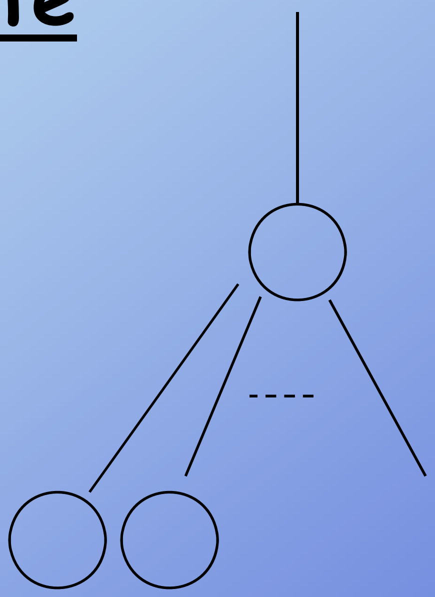
Tree structure



Tree structure



Implementation note
Nodes & arcs



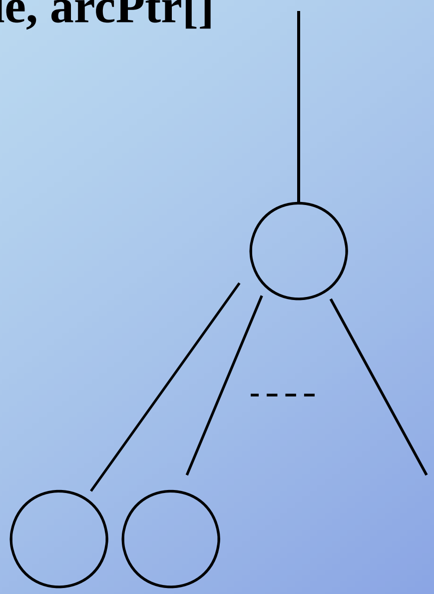
NODE
Pointer to data
Data transformation
Pointer to arcs

ARC
Transform of one next node relative to parent node
Articulation transform
Pointer to node

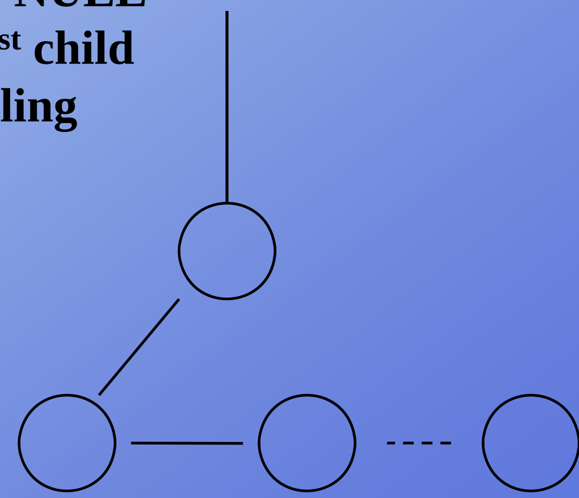
Implementation note

Representing arbitrary number of children with fixed-length data structure

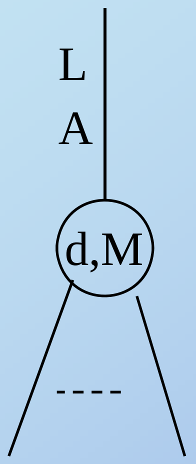
Use array of pointers
to children
In node, arcPtr[]



Node points to first child
Each child points to sibling
Last sibling points to NULL
In node: arcPtr for 1st child
In arc: arcPtr for sibling



Tree traversal



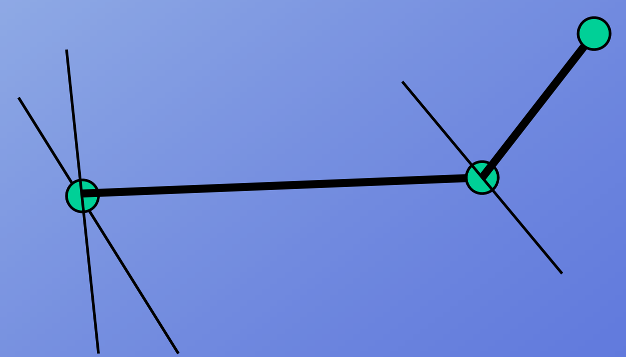

```
traverse (arcPtr,matrix)
{
        // concatenate arc matrices
        matrix = matrix*arcPtr->Lmatrix
        matrix = matrix*arcPtr->Amatrix;

        // get node and transform data
        nodePtr=acrPtr->nodePtr
        push (matrix)
        matrix = matrix * nodePTr->matrix
        aData = transformData(matrix,dataPTr)
        draw(aData)
        matrix = pop();

        // process children
        If (nodePtr->arcPtr != NULL) {
                nextArcPtr = nodePTr-> arcPtr
                while (nextArcPtr != NULL) {
                        push(matrix)
                        traverse(nextArcPtr,matrix)
                        matrix = pop()
                        nextArcPtr = nextArcPtr->arcPtr
                }
        }
}
```

OpenGL Single linkage

```
glPushMatrix();
For (i=0; i<NUMDOFS; i++) {
        glRotatef(a[i],axis[i][0], axis[i][1], axis[i][2]);
        if (linkLen[i] != 0.0) {
                draw_linkage(linkLen[i]);
                glTranslatef(0.0,linkLen[i],0.0);
        }
}
glPopMatrix();
```

OpenGL concatenates matrices

A[i] – joint angle
Axis[i] – joint axis
linkLen[i] – length of link

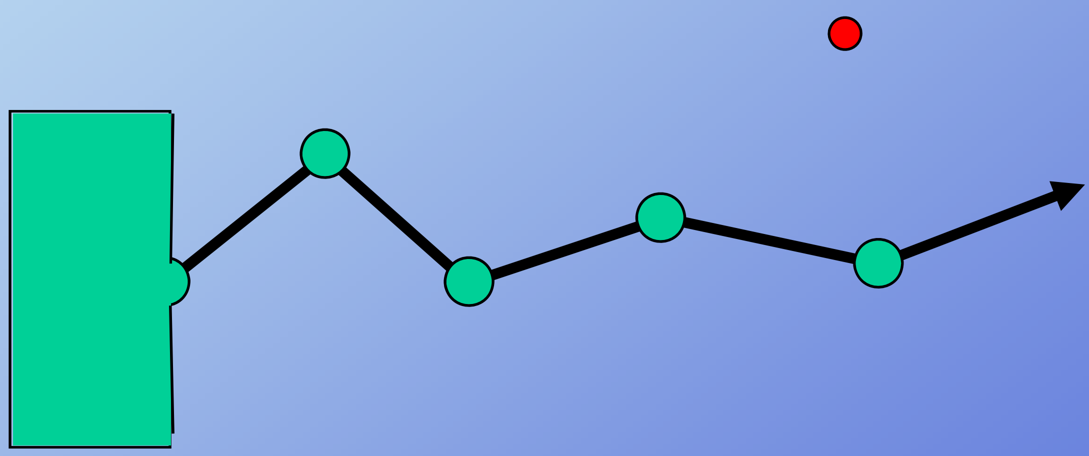


Inverse kinematics

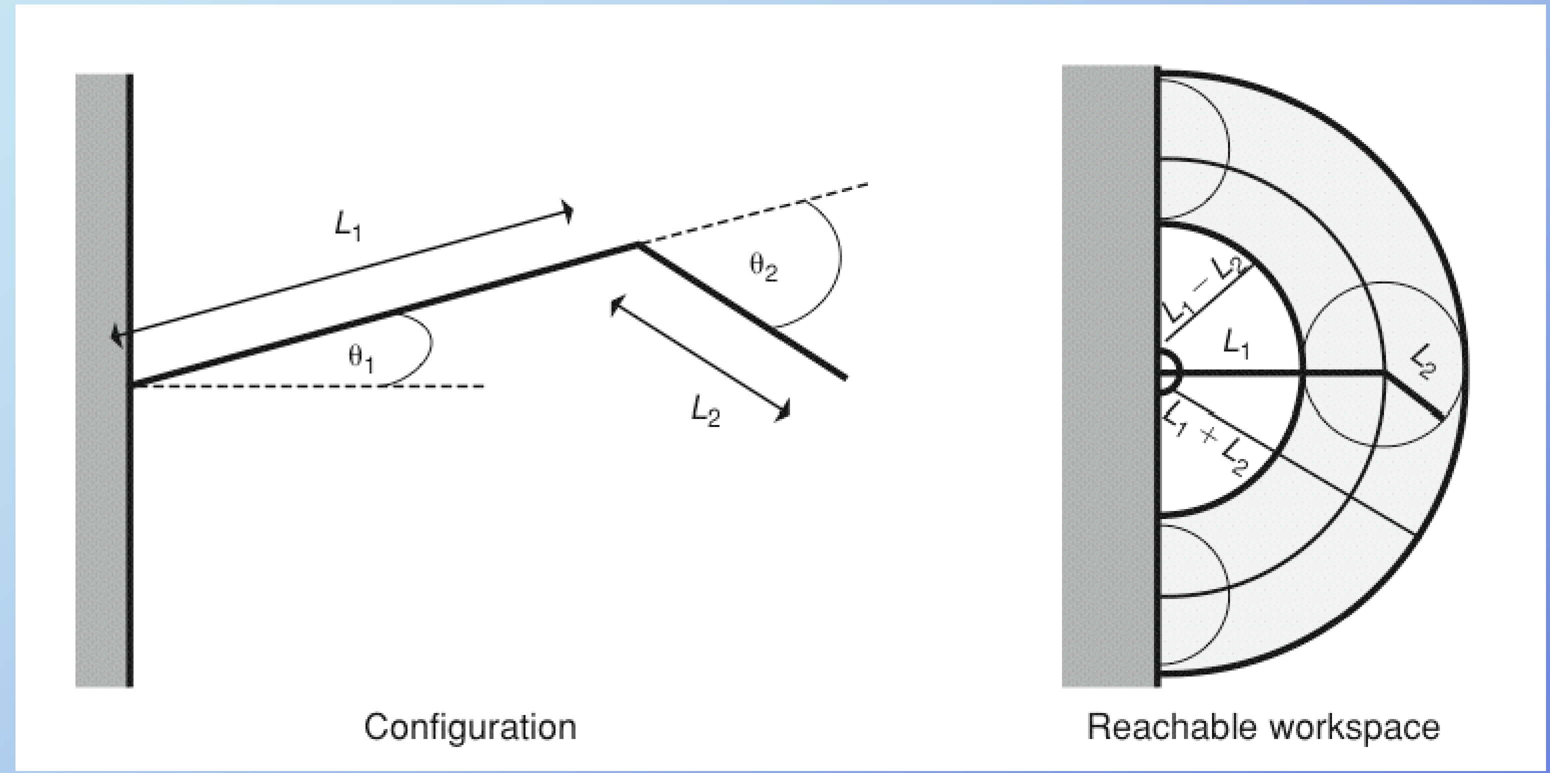
Given goal position (and orientation) for end effector

Compute internal joint angles

If simple enough => analytic solution
Else => numeric iterative solution

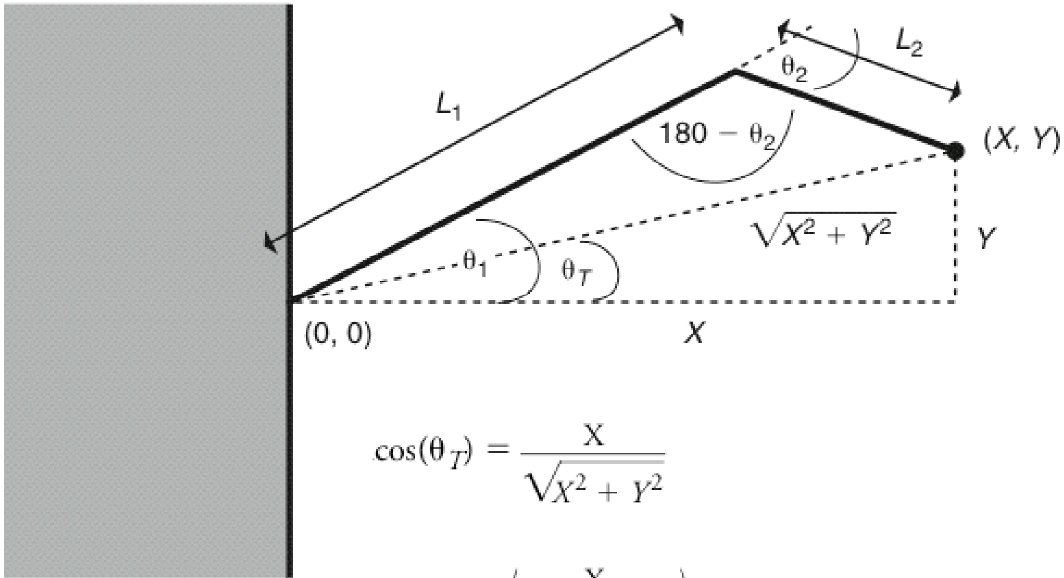


Inverse kinematics - spaces



Configuration space
Reachable workspace
Dextrous workspace

Analytic inverse kinematics



$$\cos(\theta_T) = \frac{X}{\sqrt{X^2 + Y^2}}$$

$$\theta_T = \operatorname{acos}\left(\frac{X}{\sqrt{X^2 + Y^2}}\right)$$

$$\cos(\theta_1 - \theta_T) = \frac{L_1^2 + X^2 + Y^2 - L_2^2}{2L_1\sqrt{X^2 + Y^2}} \qquad \text{(cosine rule)}$$

$$\theta_1 = \operatorname{acos}\left(\frac{L_1^2 + X^2 + Y^2 - L_2^2}{2L_1\sqrt{X^2 + Y^2}}\right) + \theta_T$$

$$\cos(180 - \theta_2) = -\cos(\theta_2) = \frac{L_1^2 + L_2^2 - (X^2 + Y^2)}{2L_1L_2} \qquad \text{(cosine rule)}$$

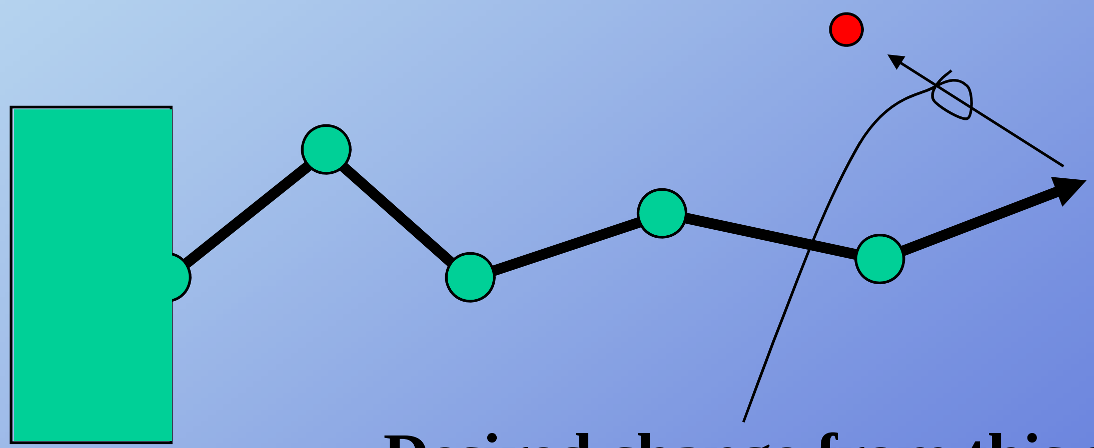
$$\theta_2 = a\cos\left(-\frac{L_1^{\ 2} + L_2^{\ 2} - (X^2 + Y^2)}{2L_1L_2}\right)$$

IK - numeric

If linkage is too complex to solve analytically

E.g., human arm is typically modeled as 3-1-3 or 3-2-2 linkage

Solve iteratively – numerically solve for step toward goal



Desired change from this specific pose

Compute set of changes to the pose to effect that change

IK math notation

$$y_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$y_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$y_3 = f_3(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$y_4 = f_4(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$y_5 = f_5(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$y_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$Y = F(X)$$

IK - chain rule

$$\frac{dy_i}{dt} = \frac{\partial f_i}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f_i}{\partial x_2}\frac{\partial x_2}{dt} + \frac{\partial f_i}{\partial x_3}\frac{\partial x_3}{dt} + \frac{\partial f_i}{\partial x_4}\frac{\partial x_4}{dt} + \frac{\partial f_i}{\partial x_5}\frac{\partial x_5}{dt} + \frac{\partial f_i}{\partial x_6}\frac{\partial x_6}{dt}$$

$$\dot{Y} = \frac{\partial F}{\partial X}\dot{X}$$

Inverse Kinematics - Jacobian

$$\dot{Y} = \frac{\partial F}{\partial X}\dot{X}$$

$$V = J(\theta)\dot{\theta}$$

Desired motion of end effector

Unknown change in articulation variables

The *Jacobian* is the matrix relating the two: it's a function of current variable values

Inverse Kinematics - Jacobian

$$V = J(\theta)\dot{\theta}$$

$$V = \lfloor \underbrace{v_x, v_y, v_z}_{\text{Change in position}}, \underbrace{\omega_x, \omega_y, \omega_z}_{\text{Change in orientation}} \rfloor$$

$$\dot{\theta} = \lfloor \underbrace{\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \dot{\theta}_4, \dot{\theta}_5, \dot{\theta}_6}_{\text{Change in articulation variables}} \rfloor$$

Change in position

Change in orientation

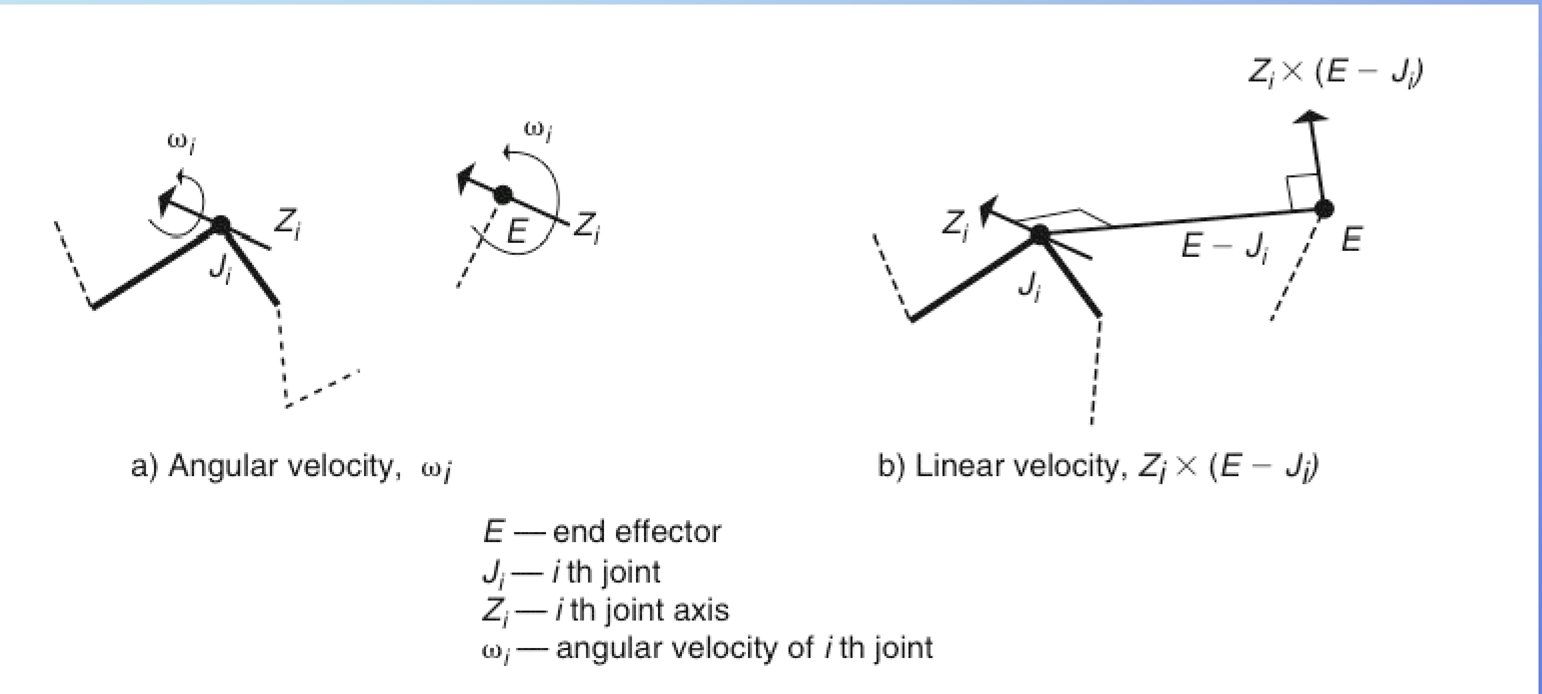
$$J = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} & \cdots & \frac{\partial p_x}{\partial \theta_6} \\ \frac{\partial p_y}{\partial \theta_1} & \cdots & & \\ \cdots & & & \\ \frac{\partial \alpha_z}{\partial \theta_1} & & & \frac{\partial \alpha_z}{\partial \theta_6} \end{bmatrix}$$

Jacobian

Change in articulation variables

IK – computing the Jacobian

(need to convert to global coordinates)

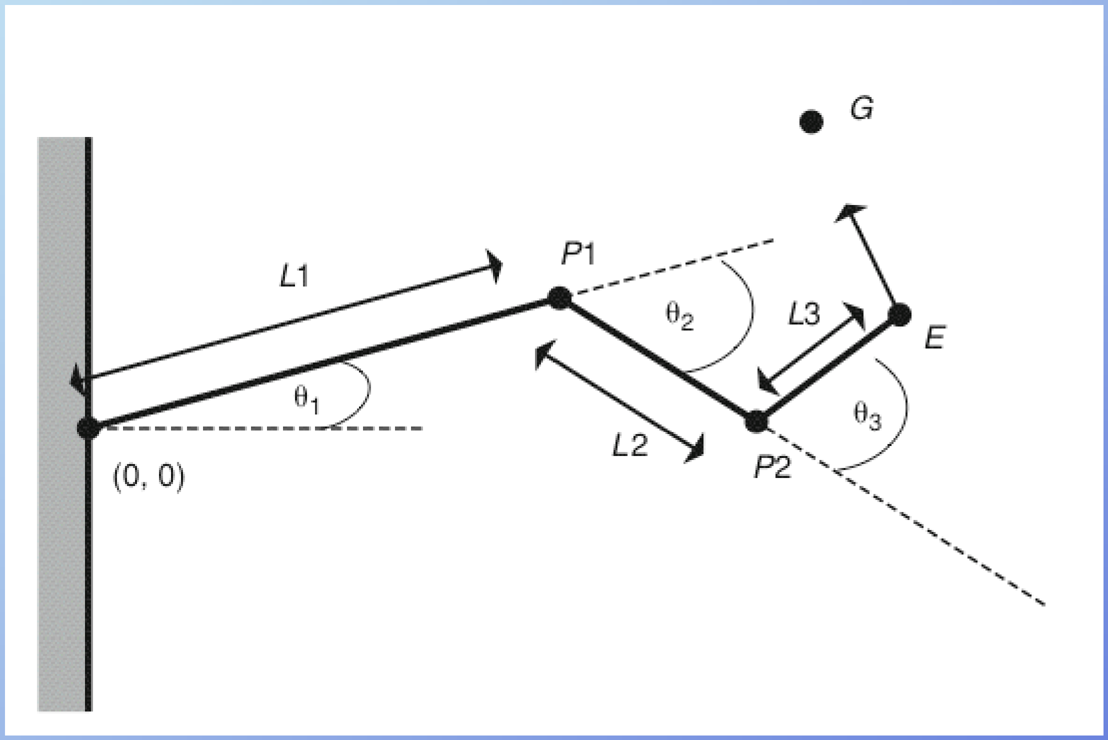


Change in orientation

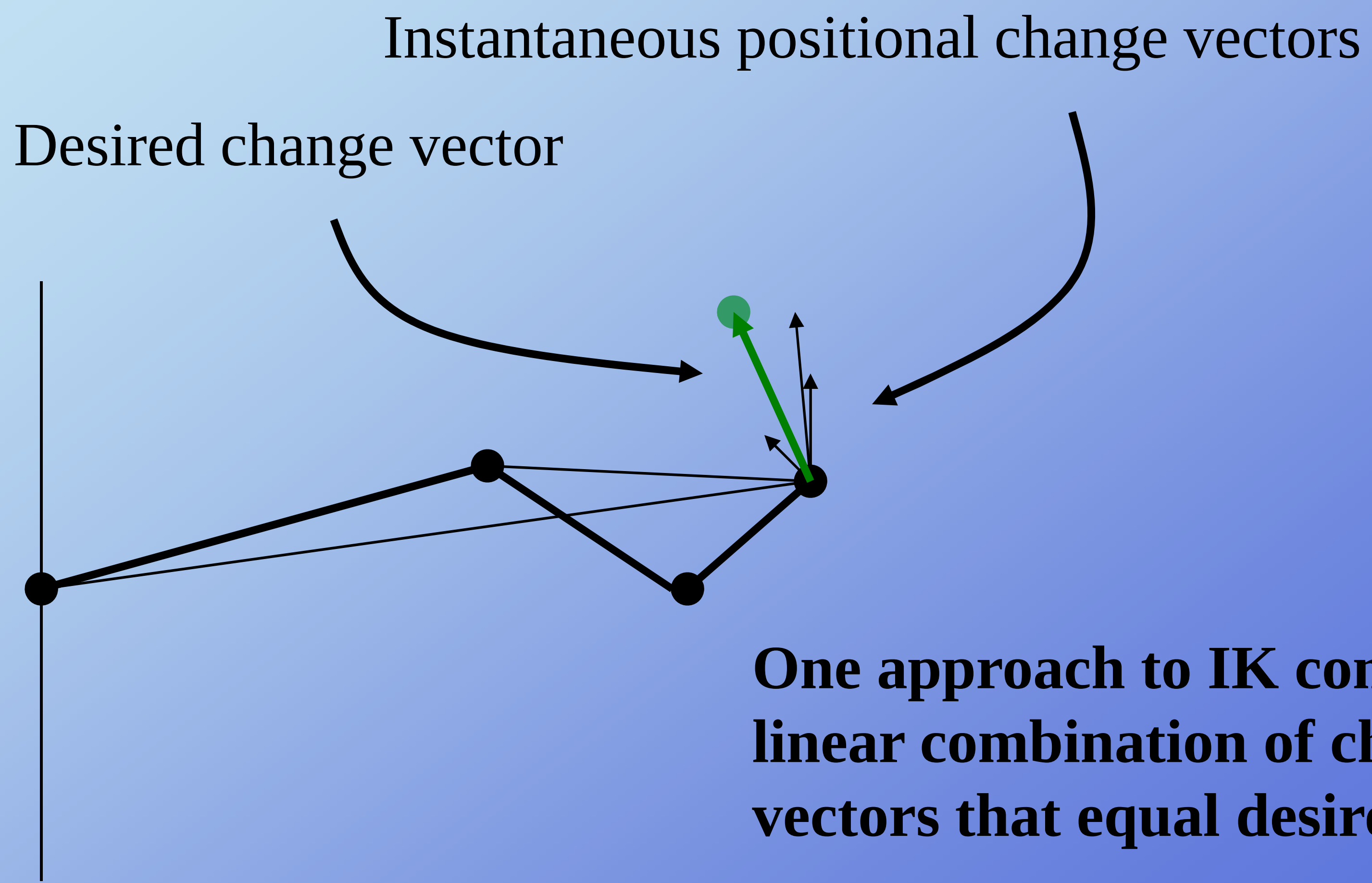
Change in position

Only valid instantaneously

IK - configuration



IK – compute positional change vectors induced by changes in joint angles



One approach to IK computes linear combination of change vectors that equal desired vector

IK - compute position and axis of joints

```
Set identity matrix

for (i=0; i<NUMDOFS; i++) {
  record_transformed_joint(i)
  glRotate(angle[i],axis[i][0],axis[i][1],axis[i][2]);
  append_rotation(angle[i],axis[i][0],axis[i][1],axis[i][2]);
  if (linkLen[i] != 0) {
    draw_linkage(linkLen[i]);
    glTranslatef(0.0,linkLen[i],0.0);
    append_translation(0,linkLen[i],0);
  }
}
record_endEffector();
```

IK - append rotation

If joint axis is:

one of major axes: 3 cases ofsimple rotation

Arbitrary axis – angle-axis to matrix conversion

IK - append translation

Form translation matrix

Matrix

Transformed coordinate system

Position

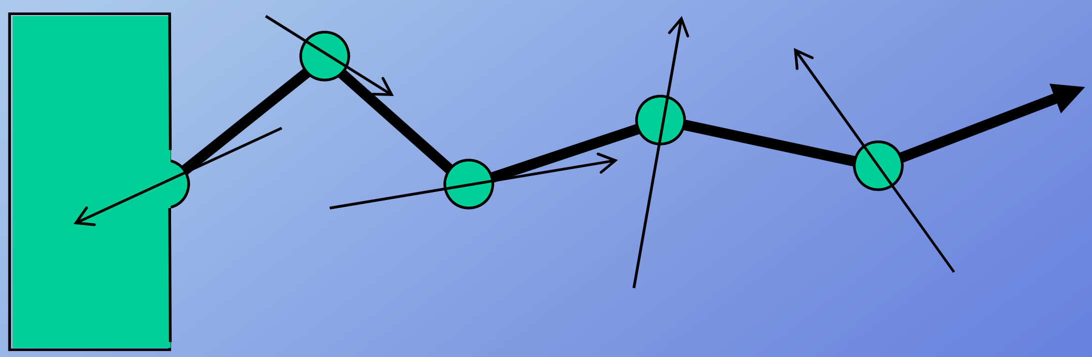
Transforms axis of rotation

IK - record joint information

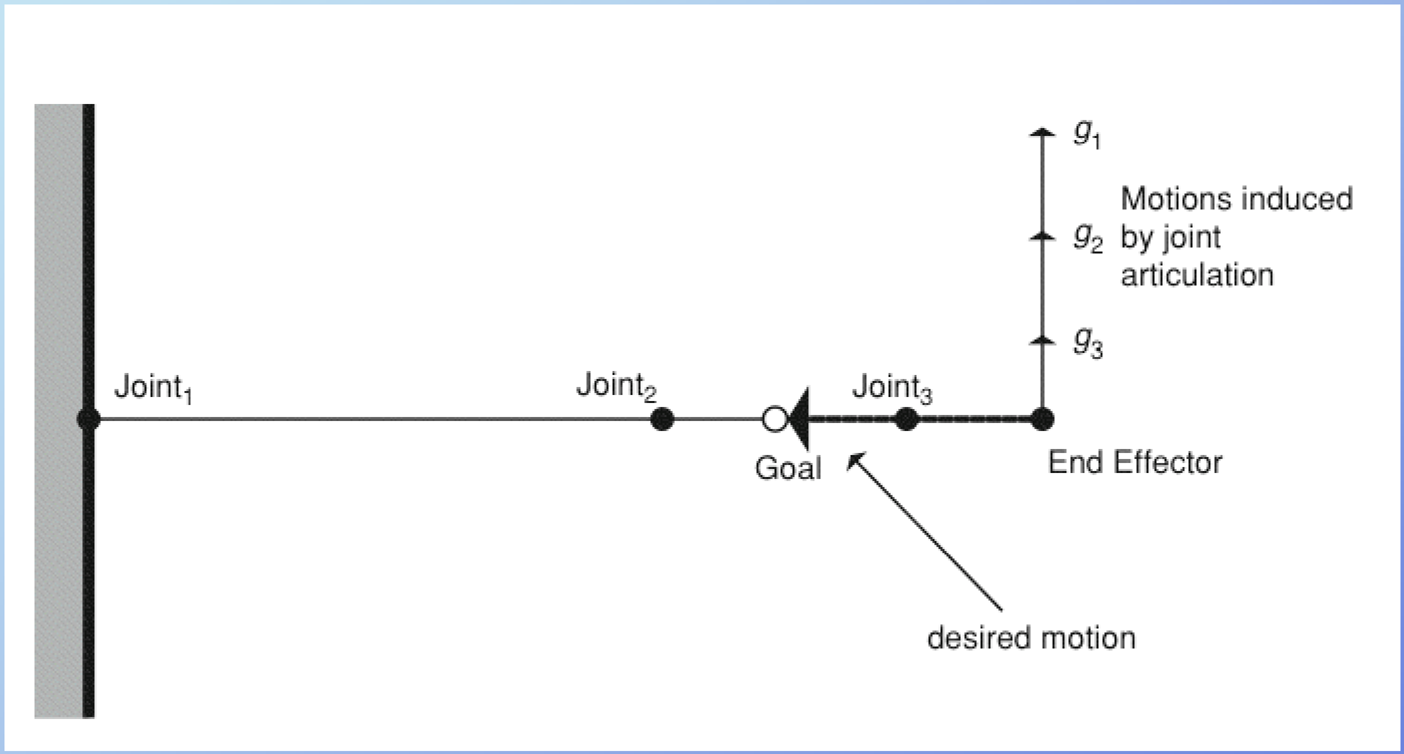
Joint position – last column of matrix

Joint coordinate system – upper left 3x3 submatrix

Joint axis – transform local joint axis vector by matrix



IK - singularity



Some singular configurations are not so easily recognizable

Near singular configurations are also problematic – why?

Inverse Kinematics - Numeric

Given
- **Current configuration**
- **Goal position/orientation**

Determine
- **Goal vector**
- **Positions & local coordinate systems of interior joints (in global coordinates)**
- **Jacobian**

$$V = J(\theta)\dot{\theta}$$ Is in same form as more recognizable : $$Ax = b$$

Solve & take small step – or clamp acceleration or clamp velocity

Repeat until:
- **Within epsilon of goal**
- **Stuck in some configuration**
- **Taking too long**

Solving

If J square, compute inverse, J^{-1}

If J not square, usually under-constrained: more DoFs than constraints
Requires use of pseudo-inverse of Jacobian

$$V = J\dot{\theta}$$

$$J^T V = J^T J\dot{\theta}$$

$$\left(J^T J\right)^{-1} J^T V = \left(J^T J\right)^{-1} J^T J\dot{\theta}$$

$$J^+ V = \dot{\theta}$$

Solving

Avoid direct computation of inverse by substitution solving Ax=B form, then substituting back

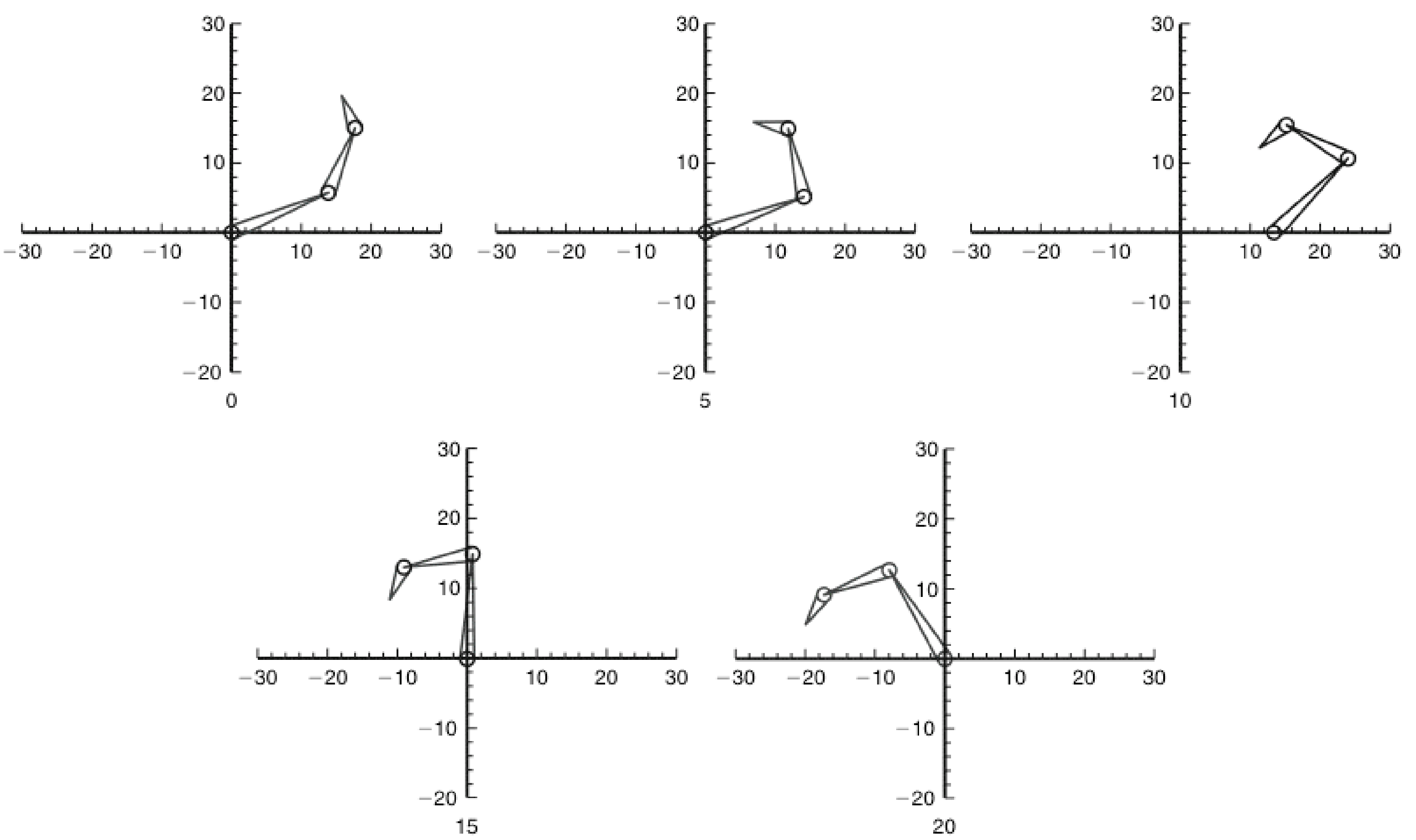
$$V = J\dot{\theta}$$

$$\boxed{J^T\beta = \dot{\theta}}$$

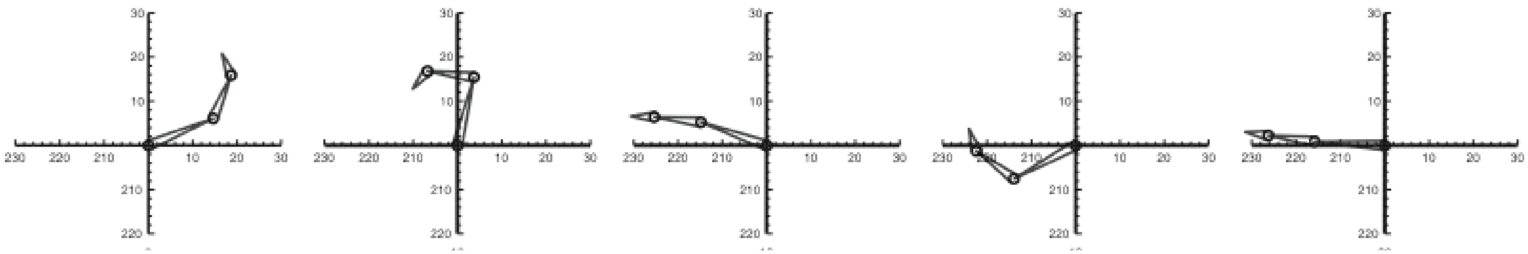
$$\boxed{V = JJ^T\beta}$$

$$J^T\beta = \dot{\theta}$$

IK – Jacobian solution



IK - Jacobian solution - problem



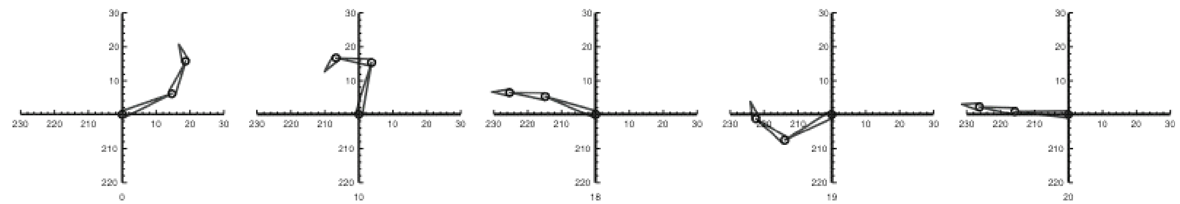
When goal is out of reach
Bizarre undulations can occur
As armature tries to reach the unreachable

→ **Add a damping factor**

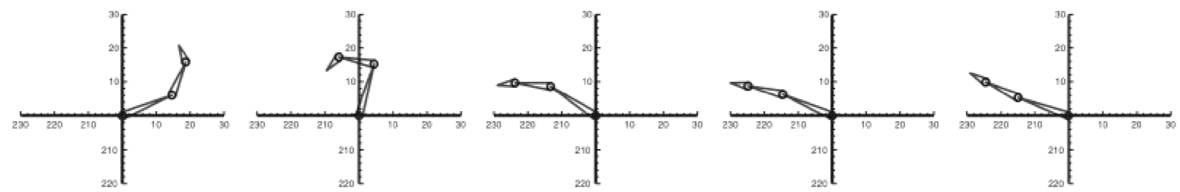
IK - Jacobian w/ damped least squares

Undamped form:

$$\dot{\theta} = \left(J^T J\right)^{-1} J^T V$$



a) The pseudoinverse of Jacobian solution without damped least squares



b) The pseudoinverse of Jacobian solution with damped least squares

Damped form with user parameter:

$$\dot{\theta} = J^T (JJ^T + \lambda^2 I)^{-1} V$$

IK – Jacobian w/ control term

Physical systems (i.e. robotics) and synthetic character simulation (e.g., human figure) have limits on joint values

IK allows joint angle to have any value

Difficult (computationally expensive) to incorporate hard constraints on joint values

Take advantage of redundant manipulators - Allow user to set parameter that urges DOF to a certain value

Does not enforce joint limit constraints, but can be used to keep joint angles at mid-range values

IK – Jacobian w/ control term

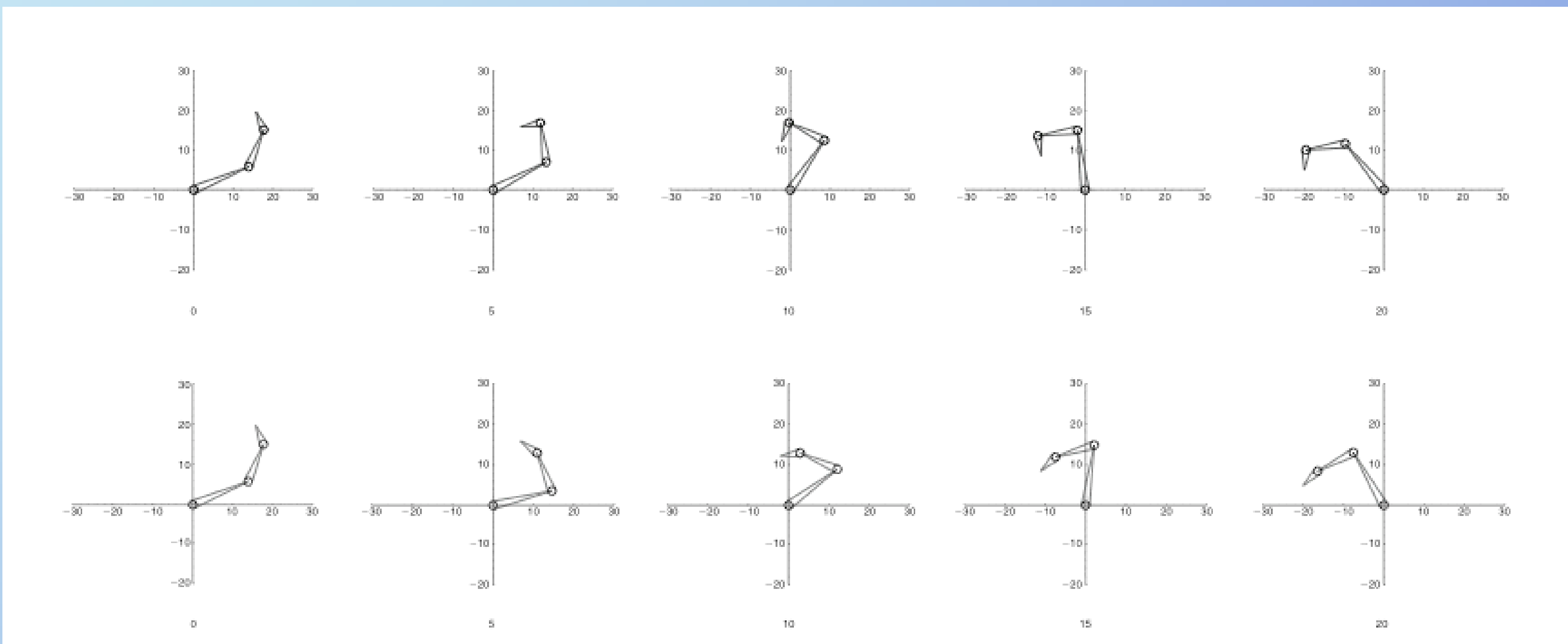
$$\dot{\theta} = J^{+}V + (J^{+}J - I)^{-1}z$$

$$z = \alpha_i(\theta_i - \theta_{ci})^2$$

$$\left\{\begin{aligned} V &= J\dot{\theta} \\ V &= J(J^{+}J - I)z \\ V &= (JJ^{+}J - J)z \\ V &= 0z \\ V &= 0 \end{aligned}\right.$$

Change to the pose parameter in the form of the control term adds nothing to the velocity

IK - Jacobian w/ control term

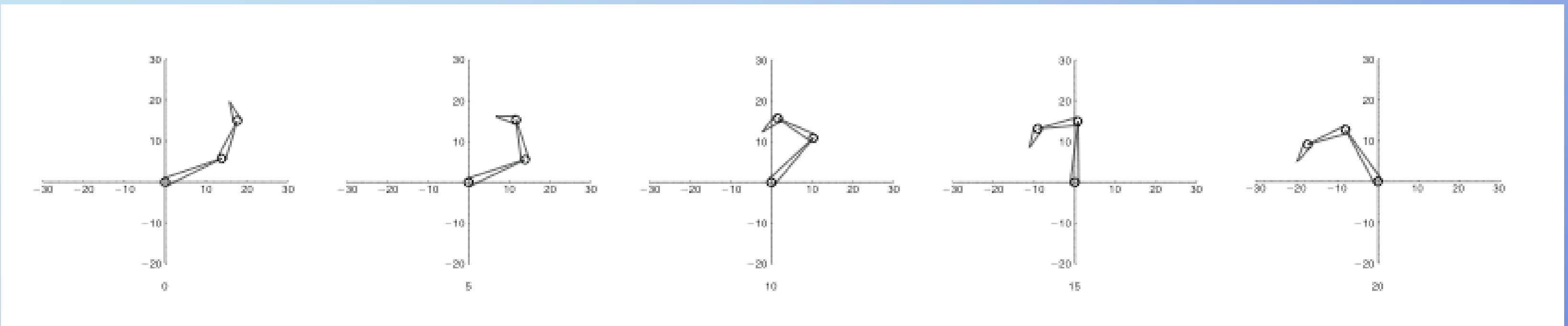


All bias to 0
Top gains = {0.1, 0.5, 0.1}
Bottom gains = {0.1, 0.1, 0.5}

$$\dot{\theta} = J^{+}V + (J^{+}J - I)^{-1} z$$

$$z = \alpha_i (\theta_i - \theta_{ci})^2$$

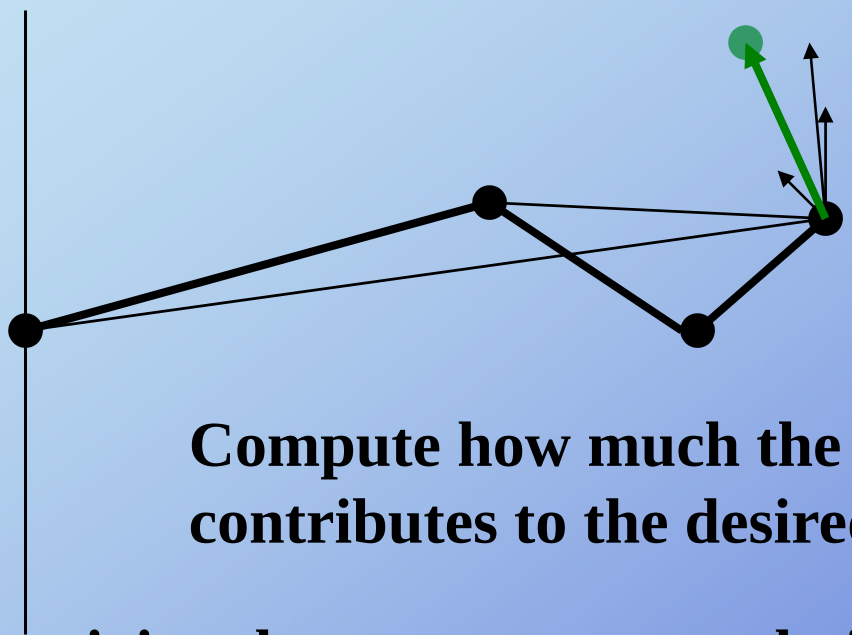
IK – alternate Jacobian



Jacobian formulated to pull the goal toward the end effector

Use same method to form Jacobian but use goal coordinates instead of end-effector coordinates

IK - Transpose of the Jacobian



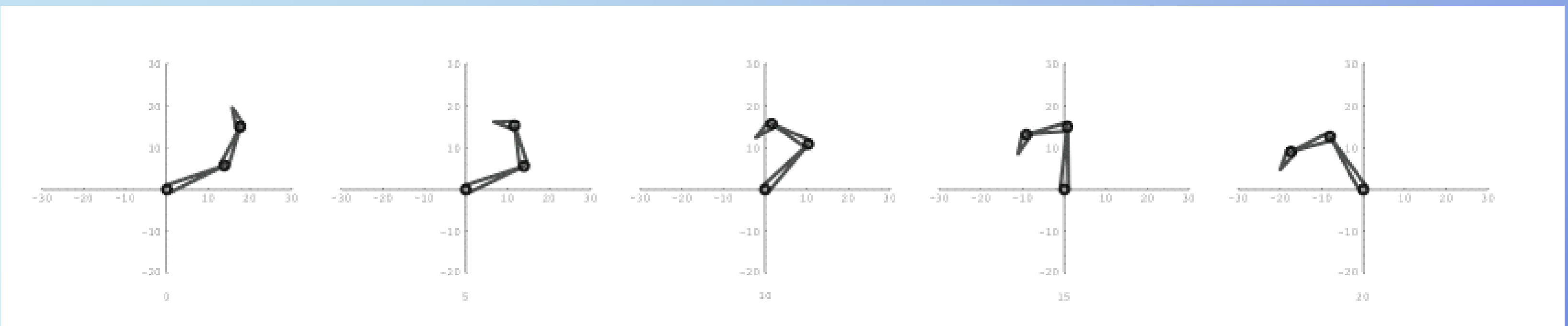
Compute how much the change vector contributes to the desired change vector:

Project joint change vector onto desired change vector

Dot product of joint change vector and desired change vector => Transpose of the Jacobian

IK – Transpose of the Jacobian

$$J^T V = \dot{\theta}$$



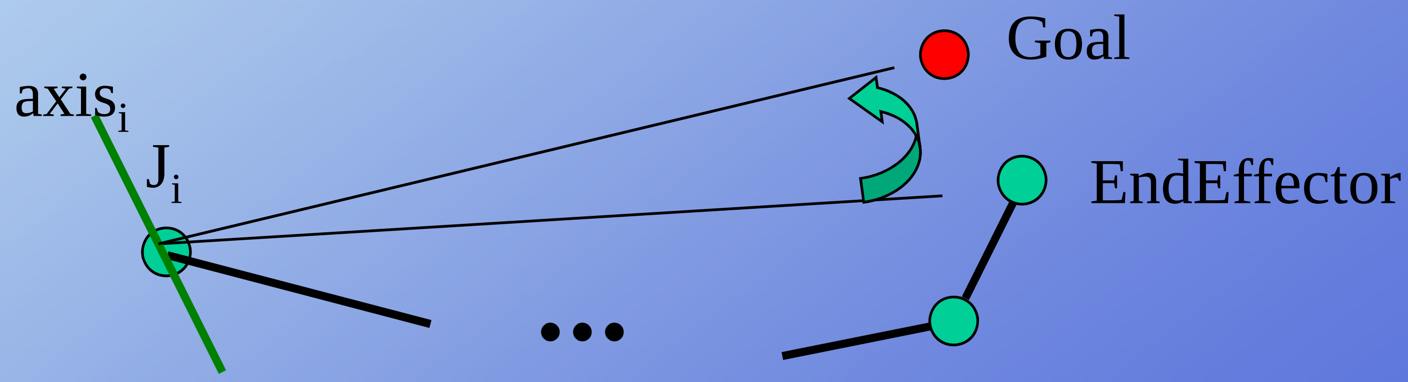
$$J^T = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_y}{\partial \theta_1} & \cdots & \frac{\partial \alpha_z}{\partial \theta_1} \\ \frac{\partial p_x}{\partial \theta_2} & \cdots & & \\ \cdots & & & \\ \frac{\partial p_x}{\partial \theta_6} & & & \frac{\partial \alpha_z}{\partial \theta_6} \end{bmatrix} \qquad V = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

IK – cyclic coordinate descent

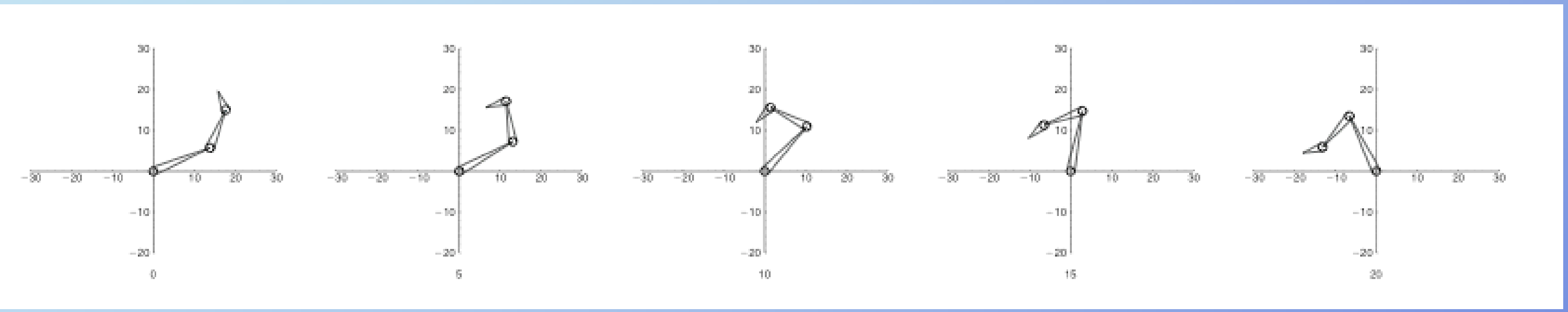
Heuristic solution

Consider one joint at a time, from outside in
At each joint, choose update that best gets end effector to goal position

In 2D – pretty simple

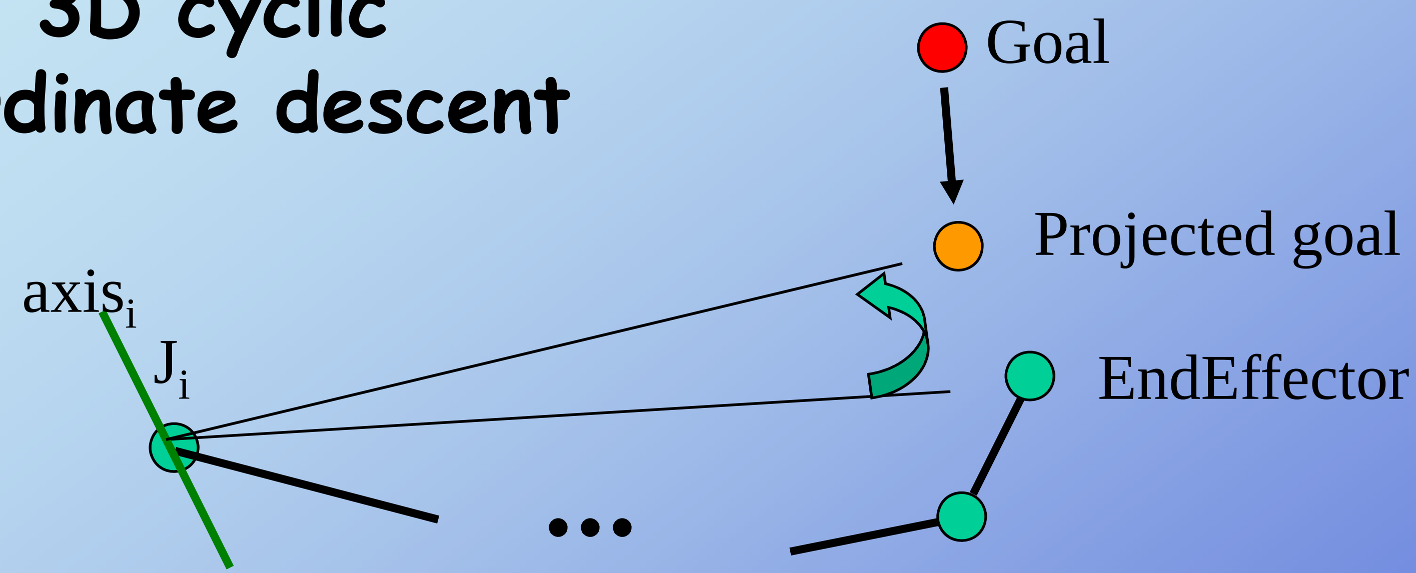


IK – cyclic coordinate descent



In 3D, a bit more computation is needed

IK – 3D cyclic coordinate descent



First – goal has to be projected onto plane defined by axis (normal to plane) and EF

Second– determine angle at joint

IK – cyclic coordinate descent – 3D

Other orderings of processing joints are possible

Because of its procedural nature

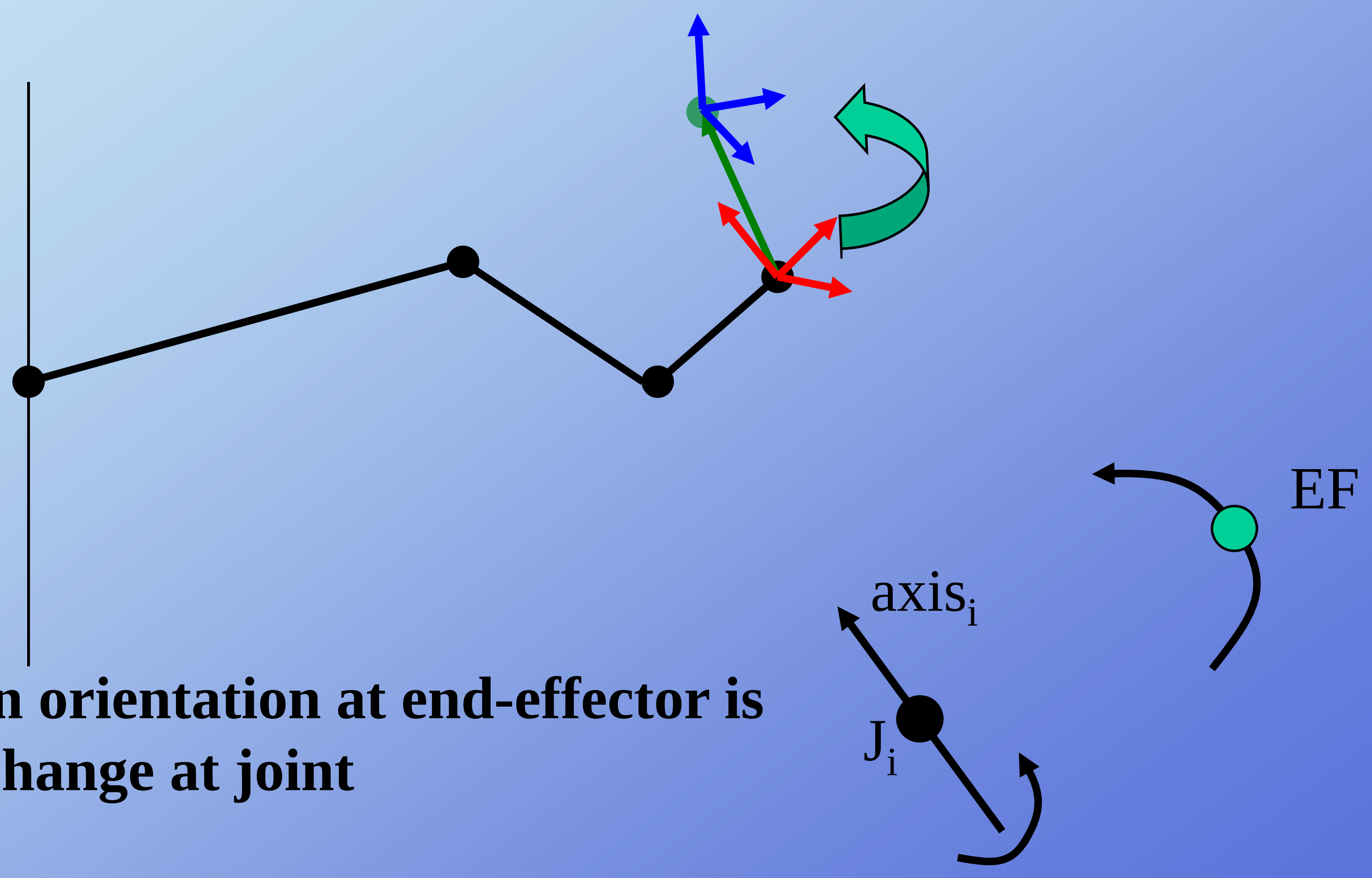
- **Lends itself to enforcing joint limits**
- **Easy to clamp angular velocity**

Inverse kinematics - review

Analytic method
Forming the Jacobian
Numeric solutions

- **Pseudo-inverse of the Jacobian**
- **J^+ with damping**
- **J^+ with control term**
- **Alternative Jacobian**
- **Transpose of the Jacobian**
- **Cyclic Coordinate Descent (CCD)**

Inverse kinematics - orientation



Change in orientation at end-effector is same as change at joint

Inverse kinematics - orientation

How to represent orientation (at goal, at end-effector)?
How to compute difference between orientations?
How to represent desired change in orientation in V vector?
How to incorporate into IK solution?

Matrix representation: M_g, M_{ef}

Difference $M_d = M_{ef}^{-1} M_g$

Use scaled axis of rotation: $\theta(a_x\ a_y\ a_z)$:

- Extract quaternion from M_d
- Extract (scaled) axis from quaternion

E.g., use Jacobian Transpose method:
Use projection of scaled joint axis onto extracted axis