Object Intersection

Implicit forms
\[ F(x,y,z) = 0 \]

Explicit forms
Analytic form \( x = F(y,z) \)
Parametric form \((x,y,z) = P(t)\)

Ray-Object Intersection

Implicit forms
\[ F(x,y,z) = 0 \]

Ray: \( P(t) = (x,y,z) = \text{source} + t*\text{direction} = s + t*v \)

Solve for \( t \):

\[ F(P(t)) = 0 \]

Ray-Sphere Intersection

Implicit form for sphere at origin of radius 1

Ray:

\[ F(x,y,z) = x^2 + y^2 + z^2 - 1 = 0 \]

Ray:

\[ P(t) = (x,y,z) = s + tv = (s_x + tv_x, s_y + tv_y, s_z + tv_z) \]

Solve:

\[ F(P(t)) = (s_x + tv_x)^2 + (s_y + tv_y)^2 + (s_z + tv_z)^2 - 1 = 0 \]

\[ = s_x^2 + s_y^2 + s_z^2 + 2(s_x v_x + s_y v_y + s_z v_z) + t^2(v_x^2 + v_y^2 + v_z^2) - 1 = 0 \]

Use quadratic equation…
Ray-Sphere Intersection

At^2 + Bt + C = 0
A = |v|^2
B = 2s·v
C = |s|^2 - 1

\[ t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \]

B^2-4AC < 0 => no intersection
= 0 => just grazes
> 0 => two hits

Axis-Aligned Cuboid
(rectangular solid, rectangular parallelepiped)

Ray equation
\[ P(t) = s + tv \]

Planar equations

Solve for intersections with planes
\[ t_{s1} = (x_1 - s_x)/v_x \]
\[ t_{s2} = (x_2 - s_x)/v_x \]
\[ t_{s1} = (y_1 - s_y)/v_y \]
...

Rectangle

For each intersection, note whether entering or leaving square side of plane: HOW?

Ray 2
Ray 1
Ray is inside after last entering and before first exiting

Ray-Plane for arbitrary plane
Generalize from axis-aligned planes to any plane:

\[ ax + by + cz + d = 0 \]
\[ n \cdot P = -d \]
Normal Vector

Given ordered sequence of points defining a polygon how do you find a normal vector for the plane?

Note: 2 normal vectors to a plane, colinear and one is the negation of the other

Ordered: e.g., clockwise when viewed from the front of the face

Right hand v. left hand space

\[ n = (V_1 - V_2) \times (V_3 - V_2) \]

Ray-Plane

\[ Ax + by + cz + d = 0 \]

\[ n \cdot P = -d \]

\[ P = s + t \cdot v \]

\[ n \cdot (s + t \cdot v) = -d \]

\[ t = -d + n \cdot s / (n \cdot v) \]
**Ray-Polyhedron**

*Polyhedron* - volume bounded by flat faces
- Each face is defined by a ring of edges
- Each edge is shared by 2 and only 2 faces

The polyhedron can be convex or concave
- Faces can be convex or concave

**Solid Modeling**

Modeling of three-dimensional solids
- Physically realizable objects
- No infinitely thin sheets, no lines
- Interior of object should ‘hold water’
  - Define a closed volume


---

**Polyhedron Classification**

- **Convex Polyhedron**
  - Only Convex Polygons
  - Triangulated
  - Quads
  - Mixed

- **Concave Polyhedron**
  - One or more Concave Polygons
  - Triangulated
  - Quads
  - Mixed

**Polygonal Solid Models**

Vertices of a face have a consistent ordering (e.g. clockwise) when viewed from the outside side of the face

- Each edge of a face is shared by one and only one other face
- Each edge appears oriented one way in one face and the other way in the other face

**Euler’s Formula**

\[ F - E + V = 2 \]
\[ F - E + V = 2 - 2P \]
Convex Polyhedron
volume bounded by finite number of infinite planes

Computing intersections is similar to cube but using ray-plane intersection and arbitrary number of planes

\[ n \cdot P = -d \]
\[ P(t) = s + t \cdot v \]

Use \( n \cdot v \) to determine Entering/exiting status

\[ n \cdot (s + t \cdot v) = -d \]
\[ n \cdot s + t \cdot (n \cdot v) = -d \]
\[ t = \frac{-d + n \cdot s}{n \cdot v} \]

Record maximum entering intersection - enterMax
Record minimum exiting intersection - exitMin
If (enterMax < exitMin) polyhedron is intersected

Concave Polyhedron
Find closest face (if any) intersected by ray

Need ray-face (ray-polygon) intersection test

Ray-Convex Polygon
Test to see if point is on ‘inside’ side of each edge

Dot product of normal
Cross product of ordered edge vector from edge source to point of intersection

\[ n \cdot (V \times E) > 0 \]
Ray-Concave Polyhedron

1. Intersect ray with plane
2. Determine if intersection point is inside of 2D polygon
   A) Convex polygon
   B) Concave polygon

Ray-concave polygon

Project plane and point of intersection to 2D plane
2D point-inside-a-polygon test
(can also be used for convex polygons)

Project to plane of 2 smallest coordinates
of normal vector

Form semi-infinite ray
and count ray-edge intersections

2D Point Inside a Convex Polygon

Semi-infinite ray test

\[ \left( p_1 < p_x \right) \& \left( p_y > p2_y \right) \] \& \left( p_y > p1_y \right) \& \left( p_x < p1_x \right)

\[ p_x \leq p1_x + (p2_x - p1_x)(p_y - p1_y)/(p2_y - p1_y) \]

Test to see if point is on 'inside' side of each edge
Need to know order (CW or CCW) of 2D vertices

\[ (p_x - p1_x)(p2_y - p1_y) - (p_y - p1_y)(p2_x - p1_x) \]}
Special Cases

Logically move ray up epsilon

2D point inside a polygon test

\[ \text{if}\ ((y < y_2) \& \& (y >= y_1)) \ || (y < y_1) \& \& (y >= y_2)) \]

Transformed objects

e.g., Ellipse is transformed sphere

Intersect ray with transformed object

Use inverse of object transformation to transform ray

Intersect transformed ray with untransformed object

Transformed objects

World space ray

\[ r(t) = x + tv \]
\[ t = [x_1, x_2, x_3, 1] \]
\[ v = [v_1, v_2, v_3, 0] \]

Object to world transform matrix

\[ M \]

Object space ray

\[ R(t)^O = M^{-t}s^O + M^{-t}v^O \]

Intersect ray with object in object space

Transform intersection point and normal back to world space

\[ p_{\text{world}}^O = MP_{\text{object}}^O \]
\[ N_{\text{world}}^O = (M^O)^T N_{\text{object}}^O \]
Ray-Cylinder

\[ P = s_1 + t_1 \cdot v_1 \]
\[ Q = s_2 + t_2 \cdot v_2 \]

At closest points

\[ (P(t_1) - Q(t_2)) \cdot v_1 = 0 \]
\[ (P(t_1) - Q(t_2)) \cdot v_2 = 0 \]

Ray-Ellipsoid

Geometric construction: all points \( p \) such that

\[ |p-a| + |p-b| = r \]

Algebraic equation – axis aligned, origin centered

\[
\begin{pmatrix}
  x/a \\
  y/b \\
  z/c
\end{pmatrix}
^2 = 1
\]

Ray-Quadric

\[ P(t) = (x, y, z) = s + tv = (s_x + tv_x, s_y + tv_y, s_z + tv_z) \]

\[ Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Gzx + Hyx + Hzy + Iz + J = 0 \]

http://en.wikipedia.org/wiki/Quadric