Divide-and-Conquer

Reading: CLRS Sections 2.3, 4.1, 4.2, 4.3, 28.2, 33.4.

CSE 6331 Algorithms

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Divide and Conquer

- Given an instance *x* of a problem, the divide-and-conquer method works as follows:

```
function DAC($x$)
    if $x$ is sufficiently small then
        solve it directly
    else
        divide $x$ into smaller subinstances $x_1, x_2, \ldots, x_k$;
        $y_i \leftarrow$ DAC($x_i$), for $1 \le i \le k$;
        $y \leftarrow \text{combine}(y_1, y_2, \ldots, y_k)$;
        return($y$)
```

Analysis of Divide-and-Conquer

- Typically, $x_1, \dots, x_k$ are of the same size, say $\lfloor n/b \rfloor$.
- In that case, the time complexity of DAC, $T(n)$, satisfies a recurrence:

$$T(n) = \begin{cases} c & \text{if } n \le n_0 \\ kT\left(\lfloor n/b \rfloor\right) + f(n) & \text{if } n > n_0 \end{cases}$$

- Where $f(n)$ is the running time of dividing x and combining y_i's.
- What is c?
- What is n_0?

Mergesort: Sort an array $A[1..n]$

- procedure mergesort$(A[i..j])$

```
    // Sort $A[i..j]$//
    if $i = j$ then return                  // base case //
    $m \leftarrow \lfloor (i+j)/2 \rfloor$
    mergesort$(A[i..m])$
                                            divide and conquer
    mergesort$(A[m+1..j])$
    merge$(A[i..m], A[m+1..j])$
```

- Initial call: mergesort$(A[1..n])$

Analysis of Mergesort

- Let $T(n)$ denote the running time of mergesorting an array of size n.
- $T(n)$ satisfies the recurrence:

$$T(n) = \begin{cases} c & \text{if } n \le 1 \\ T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + \Theta(n) & \text{if } n > 1 \end{cases}$$

- Solving the recurrence yields:

$$T(n) = \Theta(n \log n)$$

- We will learn how to solve such recurrences.

Linked-List Version of Mergesort

```
function mergesort$(i, j)$
    // Sort $A[i..j]$. Initially, $link[k] = 0, 1 \le k \le n$.//
    global $A[1..n]$, $link[1..n]$
    if $i = j$ then return$(i)$   // base case //
    $m \leftarrow \lfloor (i+j)/2 \rfloor$
    $ptr1 \leftarrow$ mergesort$(i, m)$
    $ptr2 \leftarrow$ mergesort$(m+1, j)$        } divide and conquer
    $ptr \leftarrow$ merge$(ptr1, ptr2)$
    return$(ptr)$
```

Solving Recurrences

- Suppose a function $T(n)$ satisfies the recurrence

$$T(n) = \begin{cases} c & \text{if } n \le 1 \\ 3T\left(\lfloor n/4 \rfloor\right) + n & \text{if } n > 1 \end{cases}$$

where c is a positive constant.

- Wish to obtain a function $g(n)$ such that $T(n) = \Theta\left(g(n)\right)$.
- Will solve it using various methods: Iteration Method, Recurrence Tree, Guess and Prove, and Master Method.

Iteration Method

Assume n is a power of 4. Say, $n = 4^m$. Then,

$$
\begin{aligned}
T(n) &= n + 3T(n/4) \\
&= n + 3\left[n/4 + 3T(n/16)\right] \\
&= n + 3(n/4) + 9\left[(n/16) + 3T(n/64)\right] \\
&= n + (3/4)n + (3/4)^2 n + 3^3 T(n/4^3) \\
&= n\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{m-1}\right] + 3^m T\left(\frac{n}{4^m}\right) \\
&= n\,\Theta(1) + O(n) = \Theta(n)
\end{aligned}
$$

So, $T(n) = \Theta(n \mid n \text{ a power of } 4) \Rightarrow T(n) = \Theta(n)$. (Why?)

Remark

- We have applied Theorem 7 to conclude $T(n)=\Theta(n)$ from $T(n)=\Theta(n \mid n$ a power of 4).
- In order to apply Theorem 7, $T(n)$ needs to be nondecreasing.
- It will be a homework question for you to prove that $T(n)$ is indeed nondecreasing.

Recurrence Tree

solving problems	time needed
1 of size n	
$\downarrow \; \uparrow$	n
3 of size $n/4$	
$\downarrow \; \uparrow$	$3 \cdot n/4$
3^2 of size $n/4^2$	
$\downarrow \; \uparrow$	$3^2 \cdot n/4^2$
3^3 of size $n/4^3$	
$\downarrow \; \uparrow$	$3^3 \cdot n/4^3$
$\vdots$	$\vdots$
3^{m-1} of size $n/4^{m-1}$	
$\downarrow \; \uparrow$	$3^{m-1} \cdot n/4^{m-1}$
3^m of size $n/4^m$	$3^m \cdot \Theta(1)$

Guess and Prove

- Solve $T(n) = \begin{cases} c & \text{if } n \le 1 \\ 3T\left(\lfloor n/4 \rfloor\right) + \Theta(n) & \text{if } n > 1 \end{cases}$
- First, guess $T(n) = \Theta(n)$, and then try to prove it.
- Sufficient to consider $n = 4^m$, $m = 0, 1, 2, \ldots$
- Need to prove: $c_1 4^m \le T(4^m) \le c_2 4^m$ for some c_1, c_2 and all $m \ge m_0$ for some m_0. We choose $m_0 = 0$ and prove by induction on m.
- IB: When $m = 0$, $c_1 4^0 \le T(4^0) \le c_2 4^0$ if $c_1 \le c \le c_2$.
- IH: Assume $c_1 4^{m-1} \le T(4^{m-1}) \le c_2 4^{m-1}$ for some c_1, c_2.

- IS: $T(4^m) = 3T(4^{m-1}) + \Theta(4^m)$

$$\begin{aligned} &\le 3c_2 4^{m-1} + c_2' 4^m \text{ for some constant } c_2' \\ &= \left(3c_2/4 + c_2'\right)4^m \\ &\le c_2 4^m \quad \text{if } c_2' \le c_2/4 \end{aligned}$$

$$\begin{aligned} T(4^m) &= 3T(4^{m-1}) + \Theta(4^m) \\ &\ge 3c_1 4^{m-1} + c_1' 4^m \text{ for some constant } c_1' \\ &= \left(3c_1/4 + c_1'\right)4^m \\ &\ge c_1 4^m \quad \text{if } c_1' \ge c_1/4 \end{aligned}$$

- Let c_1, c_2 be such that $c_1 \le c \le c_2$, $c_2' \le c_2/4$, $c_1/4 \le c_1'$.

Then, $c_1 4^m \le T(4^m) \le c_2 4^m$ for all $m \ge 0$.

The Master Theorem

- Definition: $f(n)$ is polynomially smaller than $g(n)$, denoted as $f(n) \ll g(n)$, iff $f(n) = O(g(n)n^{-\varepsilon})$, or $f(n)n^{\varepsilon} = O(g(n))$, for some $\varepsilon > 0$.
- For example, $1 \ll \sqrt{n} \ll n^{0.99} \ll n \ll n^{2}$.
- Is $1 \ll \log n$? Or $n \ll n \log n$?
- To answer these, ask yourself whether or not $n^{\varepsilon} = O(\log n)$.
- For convenience, write $f(n) \approx g(n)$ iff $f(n) = \Theta\big(g(n)\big)$.
- Note: the notations $\ll$ and $\approx$ are good only for this class.

The Master Theorem

If $T(n)$ satisfies the recurrence $T(n) = aT(n/b) + f(n)$, then $T(n)$ is bounded asymptotically as follows.

1. If $f(n) \ll n^{\log_b a}$, then $T(n) = \Theta\left(n^{\log_b a}\right)$.
2. If $f(n) \gg n^{\log_b a}$, then $T(n) = \Theta\left(f(n)\right)$.
3. If $f(n) \approx n^{\log_b a}$, then $T(n) = \Theta\left(f(n)\log n\right)$.
4. If $f(n) \approx n^{\log_b a} \log^k n$, then $T(n) = \Theta\left(f(n)\log n\right)$.

In case 2, it is required that $af(n/b) \le cf(n)$ for some $c < 1$, which is satisfied by most $f(n)$ that we shall encounter.

In the theorem, n/b should be interpreted as $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$.

Examples: solve these recurrences

- $T(n) = 3T(n/4) + n.$
- $T(n) = 9T(n/3) + n.$
- $T(n) = T(2n/3) + 1.$
- $T(n) = 3T(n/4) + n\log n.$
- $T(n) = 7T(n/2) + \Theta(n^2).$
- $T(n) = 2T(n/2) + n\log n.$
- $T(n) = T(n/3) + T(2n/3) + n.$

$$T(n) = aT(n/b) + f(n)$$

solving problems	time needed
1 of size n	
$\downarrow \; \uparrow$	$f(n)$
a of size n/b	
$\downarrow \; \uparrow$	$a \cdot f(n/b)$
a^2 of size n/b^2	
$\downarrow \; \uparrow$	$a^2 \cdot f(n/b^2)$
$\vdots$	$\vdots$
$a^{\log_b n-1}$ of size $n/b^{\log_b n-1}$	
$\downarrow \; \uparrow$	$a^{\log_b n-1} \cdot f(n/b^{\log_b n-1})$
$a^{\log_b n}$ of size $n/b^{\log_b n}$	$a^{\log_b n} \cdot \Theta(1)$

$$T(n) = aT(n/b) + f(n)$$

$$1 \text{ of size } n \qquad\qquad f(n)$$

$$\downarrow \ \uparrow$$

$$a \text{ of size } n/b \qquad\qquad a \cdot f(n/b)$$

$$\downarrow \ \uparrow$$

$$a^2 \text{ of size } n/b^2 \qquad\qquad a^2 \cdot f(n/b^2)$$

$$\downarrow \ \uparrow$$

$$\vdots \qquad\qquad \vdots$$

$$a^{\log_b n-1} \text{ of size } n/b^{\log_b n-1} \qquad\qquad a^{\log_b n-1} \cdot f(n/b^{\log_b n-1})$$

$$\downarrow \ \uparrow$$

$$a^{\log_b n} \text{ of size } n/b^{\log_b n} \qquad\qquad a^{\log_b n} \cdot \Theta(1)$$

$$T(n) = \sum_{i=0}^{\log_b n-1} a^i f\left(n/b^i\right) + n^{\log_b a}$$

(Note: $\log_b n = \dfrac{\log_a n}{\log_a b} = \log_a n \cdot \log_b a$)

Suppose $f(n) = \Theta\left(n^{\log_b a}\right)$.

Then $f\left(n/b^i\right) = \Theta\left(\left(n/b^i\right)^{\log_b a}\right) = \Theta\left(\frac{n^{\log_b a}}{b^{i \log_b a}}\right) = \Theta\left(\frac{n^{\log_b a}}{a^i}\right)$,

and thus $a^i f\left(n/b^i\right) = \Theta\left(n^{\log_b a}\right)$. Then, we have

$$T(n) = \sum_{i=0}^{\log_b n - 1} a^i f\left(n/b^i\right) + n^{\log_b a} \quad \text{(from the previous slide)}$$

$$= \Theta\left(\sum_{i=0}^{\log_b n - 1} n^{\log_b a} + n^{\log_b a}\right) = \Theta\left(n^{\log_b a} \log n\right) = \Theta\left(f(n) \log n\right)$$

When recurrences involve roots

- Solve $T(n) = \begin{cases} 2T\left(\sqrt{n}\right) + \log n & \text{if } n > 2 \\ c & \text{otherwise} \end{cases}$
- Suffices to consider only powers of 2. Let $n = 2^m$.
- Define a new function $S(m) = T(2^m) = T(n)$.
- The above recurrence translates to

$$S(m) = \begin{cases} 2S\left(m/2\right) + m & \text{if } m > 1 \\ c & \text{otherwise} \end{cases}$$

- By Master Theorem, $S(m) = \Theta(m \log m)$.
- So, $T(n) = \Theta(\log n \log \log n)$

Strassen's Algorithm for Matrix Multiplication

- Problem: Compute $C = AB$, given $n \times n$ matrices A and B.
- The straightforward method requires $\Theta(n^3)$ time, using the formula $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.
- Toward the end of 1960s, Strassen showed how to multiply matrices in $O(n^{\log 7}) = O(n^{2.81})$ time.
- For $n = 100$, $n^{2.81} \approx 416{,}869$, and $n^3 = 1{,}000{,}000$.
- The time complexity was reduced to $O(n^{2.521813})$ in 1979, to $O(n^{2.521801})$ in 1980, and to $O(n^{2.376})$ in 1986.
- In the following discussion, n is assumed to be a power of 2.

- Write

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C=\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

  where each A_{ij}, B_{ij}, C_{ij} is a $n/2 \times n/2$ matrix.

- Then $$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$
- If we compute $C_{ij}=A_{i1}B_{1j}+A_{i2}B_{2j}$, the running time $T(n)$ will satisfy the recurrence $T(n)=8T(n/2)+\Theta(n^2)$
- $T(n)$ will be $\Theta(n^3)$, not better than the straightforward one.
- Good for parallel processing. What's the running time using $\Theta(n^3)$ processors?

- Strassen showed

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} M_2 + M_3 & M_1 + M_2 + M_5 + M_6 \\ M_1 + M_2 + M_4 - M_7 & M_1 + M_2 + M_4 + M_5 \end{pmatrix}$$

where

$$\begin{aligned}
M_1 &= (A_{21} + A_{22} - A_{11}) \times (B_{22} - B_{12} + B_{11}) \\
M_2 &= A_{11} \times B_{11} \\
M_3 &= A_{12} \times B_{21} \\
M_4 &= (A_{11} - A_{21}) \times (B_{22} - B_{12}) \\
M_5 &= (A_{21} + A_{22}) \times (B_{12} - B_{11}) \\
M_6 &= (A_{12} - A_{21} + A_{11} - A_{22}) \times B_{22} \\
M_7 &= A_{22} \times (B_{11} + B_{22} - B_{12} - B_{21})
\end{aligned}$$

- $T(n) = 7T(n/2) + \Theta(n^2) \Rightarrow T(n) = \Theta(n^{\log 7})$

The Closest Pair Problem

- Problem Statement: Given a set of n points in the plane, $A = \{(x_i, y_i) : 1 \le i \le n\}$, find two points in A whose distance is smallest among all pairs.
- Straightforward method: $\Theta(n^2)$.
- Divide and conquer: $O(n \log n)$.

The Divide-and-Conquer Approach

1. Partition A into two sets: $A = B \cup C$.
2. Find a closest pair $(p_1,\ q_1)$ in B.
3. Find a closest pair $(p_2,\ q_2)$ in C.
4. Let $\delta = \min\{\text{dist}(p_1,\ q_1),\ \text{dist}(p_2,\ q_2)\}$.
5. Find a closest pair $(p_3,\ q_3)$ between B and C with distance less than δ, if such a pair exists.
6. Return the pair of the three which is closest.

- Question: What would be the running time?
- Desired: $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$.

- Now let's see how to implement each step.
- Trivial: steps 2, 3, 4, 6.

Step 1: Partition A into two sets: $A = B \cup C$.

- A natural choice is to draw a vertical line to divide the points into two groups.
- So, sort $A[1..n]$ by x-coordinate. (Do this only once.)
- Then, we can easily partition any set $A[i..j]$ by

$$A[i..j] = A[i..m] \cup A[m+1..j]$$

where $m = \lfloor (i+j)/2 \rfloor$.

Step 5: Find a closest pair between B and C with distance less than δ, if exists.

- We will write a procedure

  Closest-Pair-Between-Two-Sets$\left(A[i..j],\ ptr,\ \delta,\ (p3,\ q3)\right)$

  which finds a closest pair between $A[i..m]$ and $A[m+1..j]$ with distance less than δ, if exists.
- The running time of this procedure must be no more than $O\left(\left|A[i..j]\right|\right)$ in order for the final algorithm to be $O\left(n\log n\right)$.

Data Structures

- Let the coordinates of the n points be stored in $X[1..n]$ and $Y[1..n]$.
- For simplicity, let $A[i] = (X[i],\ Y[i])$.
- For convenience, introduce two dummy points: $A[0] = (-\infty, -\infty)$ and $A[n+1] = (\infty, \infty)$
- We will use these two points to indicate "no pair" or "no pair closer than δ."
- Introduce an array Link[1..n], initialized to all 0's.

Main Program

- Global variable: $A[0..n+1]$
- Sort $A[1..n]$ such that $X[1] \le X[2] \le \cdots \le X[n]$.

  That is, sort the given n points by x-coordinate.
- Call Procedure Closest-Pair with appropriate parameters.

Procedure Closest-Pair$\left(A[i..j],\ (p,\ q)\right)$ //Version 1//

```
 {*returns a closest pair $(p,\ q)$ in $A[i..j]$*}
• If $j-i=0$: $(p,\ q)\leftarrow(0,\ n+1)$;
• If $j-i=1$: $(p,\ q)\leftarrow(i,\ j)$;
• If $j-i>1$: $m\leftarrow\lfloor (i+j)/2\rfloor$
      Closest-Pair$\left(A[i..m],\ (p_1,\ q_1)\right)$
      Closest-Pair$\left(A[m+1..j],\ (p_2,\ q_2)\right)$
      $ptr\leftarrow$ mergesort $A[i..j]$ by $y$-coordinate into a linked list
      $\delta\leftarrow\min\left\{\mathrm{dist}(p_1,q_1),\ \mathrm{dist}(p_2,q_2)\right\}$
      Closest-Pair-Between-Two-Sets$\left(A[i..j],\ ptr,\ \delta,\ (p_3,\ q_3)\right)$
      $(p,\ q)\leftarrow$ closest of the three $(p_1,\ q_1)$, $(p_2,\ q_2)$, $(p_3,\ q_3)$
```

Time Complexity of version 1

- Initial call: $\text{Closest-Pair}\left(A[1..n],\ (p,\ q)\right)$.
- Assume Closest-Pair-Between-Two-Sets needs $\Theta(n)$ time.
- Let $T(n)$ denote the worst-case running time of $\text{Closest-Pair}\left(A[1..n],\ (p,\ q)\right)$.
- Then, $T(n) = 2T(n/2) + \Theta(n \log n)$.
- So, $T(n) = \Theta\left(n \log^2 n\right)$.
- Not as good as desired.

How to reduce the time complexity to $O(n \log n)$?

- Suppose we use Mergesort to sort $A[i..j]$:

  $ptr \leftarrow$ Sort $A[i..j]$ by y-coordinate into a linked list

- Rewrite the procedure as version 2.
- We only have to sort the base cases and perform "merge."
- Here we take a free ride on Closest-Pair for dividing.
- That is, we combine Mergesort with Closest-Pair.

```
Procedure Closest-Pair$(A[i..j], (p, q))$ //Version 2//
• If $j-i=0$: $(p, q) \leftarrow (0, n+1)$;
• If $j-i=1$: $(p, q) \leftarrow (i, j)$;
• If $j-i>1$: $m \leftarrow \lfloor (i+j)/2 \rfloor$
      Closest-Pair$(A[i..m], (p_1, q_1))$
      Closest-Pair$(A[m+1..j], (p_2, q_2))$
      $\left.\begin{array}{l} ptr1 \leftarrow \text{Mergesort}(A[i..m]) \\ ptr2 \leftarrow \text{Mergesort}(A[m+1..j]) \\ ptr \leftarrow \text{Merge}(ptr1, ptr2) \end{array}\right\} \text{mergesort}(A[i..\text{j}])$
    (the rest is the same as in version 1)
```

```
Procedure Closest-Pair$\left(A[i..j],\ (p,\ q),\ ptr\right)$ //final version//
  {*mergesort $A[i..j]$ by $y$ and find a closest pair $(p,\ q)$ in $A[i..j]$*}
  • if $j-i=0$: $(p,\ q)\leftarrow(0,\ n+1)$; $ptr\leftarrow i$
  • if $j-i=1$: $(p,\ q)\leftarrow(i,\ j)$;
        if $Y[i]\le Y[j]$ then $\{ptr\leftarrow i;\ Link[i]\leftarrow j\}$
                          else $\{ptr\leftarrow j;\ Link[j]\leftarrow i\}$
  • if $j-i>1$: $m\leftarrow\lfloor(i+j)/2\rfloor$
        Closest-Pair$\left(A[i..m],\ (p_1,\ q_1),\ ptr1\right)$
        Closest-Pair$\left(A[m+1..j],\ (p_2,\ q_2),\ ptr2\right)$
        $ptr\leftarrow$ Merge$\left(ptr1,\ ptr2\right)$
        (the rest is the same as in version 1)
```

Time Complexity of the final version

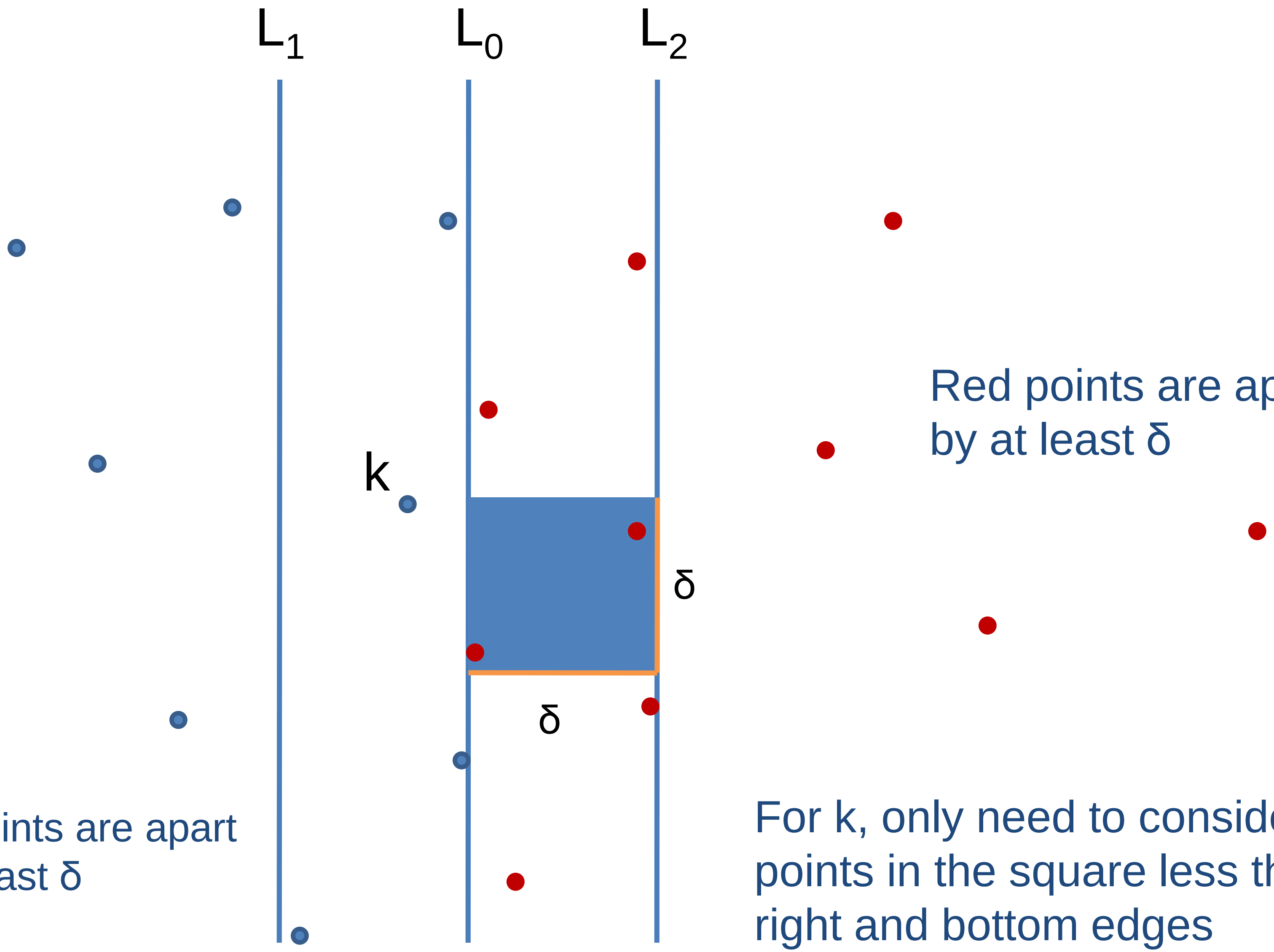
- Initial call: $\text{Closest-Pair}\left(A[1..n],\ (p,\ q),\ pqr\right)$.
- Assume Closest-Pair-Between-Two-Sets needs $\Theta(n)$ time.
- Let $T(n)$ denote the worst-case running time of $\text{Closest-Pair}\left(A[1..n],\ (p,\ q),\ pqr\right)$.
- Then, $T(n) = 2T(n/2) + \Theta(n)$.
- So, $T(n) = \Theta\left(n \log n\right)$.
- Now, it remains to write the procedure $\text{Closest-Pair-Between-Two-Sets}\left(A[i..j],\ ptr,\ \delta,\ (p_3,\ q_3)\right)$

Closest-Pair-Between-Two-Sets

- Input: $\left(A[i..j],\ ptr,\ \delta\right)$
- Output: a closest pair $(p,\ q)$ between $B = A[i..m]$ and $C = A[m+1..j]$ with distance $< \delta$, where $m = \lfloor (i+j)/2 \rfloor$. If there is no such a pair, return the dummy pair $(0,\ n+1)$.
- Time complexity desired : $O\left(\left|A[i..j]\right|\right)$.
- For each point $b \in B$, we will compute $\text{dist}(b, c)$ for $O(1)$ points $c \in C$. Similarly for each point $c \in C$.
- Recall that $A[i..j]$ has been sorted by y. We will follow the sorted linked list and look at each point.

Closest-Pair-Between-Two-Sets

- L_0: vertical line passing through the point $A[m]$.
- L_1 and L_2: vertical lines to the left and right of L_0 by δ.
- We observe that:
  - We only need to consider those points between L_1 and L_2.
  - For each point k in $A[i..m]$, we only need to consider the points in $A[m+1..j]$ that are inside the square of $\delta \times \delta$.
  - There are at most three such points.
  - And they are among the most recently visited three points of $A[m+1..j]$ lying between L_0 and L_2.
  - Similar argument for each point k in $A[m+1..j]$.



Red points are apart by at least δ

Blue points are apart by at least δ

For k, only need to consider the points in the square less the right and bottom edges

```
Closest-Pair-Between-Two-Sets($A[i..j]$, $ptr$, $\delta$, $(p_3, q_3)$)
// Find the closest pair between $A[i..m]$ and $A[m+1..j]$
   with dist $< \delta$.  If there exists no such a pair, then return
   the dummy pair $(0, n+1)$. //
   global $X[0..n+1]$, $Y[0..n+1]$, $Link[1..n]$

   $(p_3, q_3) \leftarrow (0, n+1)$
   $b_1, b_2, b_3 \leftarrow 0$   //most recently visited 3 points btwn $L_0, L_1$//
   $c_1, c_2, c_3 \leftarrow n+1$   //such points between $L_0, L_2$//
   $m \leftarrow \lfloor (i+j)/2 \rfloor$
   $k \leftarrow ptr$
```

```
while $k \neq 0$ do   //follow the linked list until end//
  1. if $|X[k] - X[m]| < \delta$ then   // consider only btwn $L_1$, $L_2$//
        if $k \le m$ then   //point $k$ is to the left of $L_0$//
            compute $d \leftarrow \min\{\text{dist}(k, c_i): 1 \le i \le 3\}$;
            if $d < \delta$ then update $\delta$ and $(p_3, q_3)$;
            $b_3 \leftarrow b_2$;   $b_2 \leftarrow b_1$;   $b_1 \leftarrow k$;
        else   //point $k$ is to the right of $L_0$//
            compute $d \leftarrow \min\{\text{dist}(k, \text{b}_i): 1 \le i \le 3\}$;
            if $d < \delta$ then update $\delta$ and $(p_3, q_3)$;
            $c_3 \leftarrow c_2$;   $c_2 \leftarrow c_1$;   $c_1 \leftarrow k$;
  2. $k \leftarrow Link[k]$
```

Convex Hull

- Problem Statement: Given a set of n points in the plane,

  say, $A = \left\{ p_1,\ p_2, p_3,\ \ldots, p_n \right\},$

  we want to find the convex hull of A.
- The convex hull of A, denoted by $CH(A)$, is the smallest convex polygon that encloses all points of A.
- Observation: segment $\overline{p_i p_j}$ is an edge of $CH(A)$ if all other points of A are on the same side of $\overline{p_i p_j}$ (or on $\overleftrightarrow{p_i p_j}$).
- Straightforward method: $\Omega(n^2)$.
- Divide and conquer: $O(n \log n)$.

Divide-and-Conquer for Convex Hull

0. Assume all x-coordinates are different, and no three points are colinear. (Will be removed later.)
1. Let A be sorted by x-coordinate.
2. If $|A| \leq 3$, solve the problem directly. Otherwise, apply divide-and-conquer as follows.
3. Break up A into $A = B \cup C$.
4. Find the convex hull of B.
5. Find the convex hull of C.
6. Combine the two convex hulls by finding the upper and lower bridges to connect the two convex hulls.

Upper and Lower Bridges

- The upper bridge between $CH(B)$ and $CH(C)$ is the the edge $\overline{vw}$, where $v \in CH(B)$ and $w \in CH(C)$, such that
  - all other vertices in $CH(B)$ and $CH(C)$ are below $\overleftrightarrow{vw}$, or
  - the two neighbors of v in $CH(B)$ and the two neighbors of w in $CH(C)$ are below $\overleftrightarrow{vw}$, or
  - the counterclockwise-neighbor of v in $CH(B)$ and the clockwise-neighbor of w in $CH(C)$ are below $\overleftrightarrow{vw}$, if v and w are chosen as in the next slide.
- Lower bridge: similar.

Finding the upper bridge

- $v \leftarrow$ the rightmost point in $CH(B)$;
  $w \leftarrow$ the leftmost point in $CH(C)$.
- Loop

```
    if counterclockwise-neighbor($v$) lies above line $\overleftrightarrow{vw}$ then
        $v \leftarrow$ counterclockwise-neighbor($v$)
    else if clockwise-neighbor($w$) lies above $\overleftrightarrow{vw}$ then
        $w \leftarrow$ clockwise neighbor($w$)
    else  exit from the loop
```

- $\overline{vw}$ is the upper bridge.

Data Structure and Time Complexity

- What data structure will you use to represent a convex hull?
- Using your data structure, how much time will it take to find the upper and lower bridges?
- What is the over all running time of the algorithm?

- We assumed:

  (1) no two points in *A* share the same *x*-coordinate

  (2) no three points in *A* are colinear
- Now let's remove these assumptions.

Orientation of three points

- Three points: $p_1(x_1, y_1),\ p_2(x_2, y_2),\ p_3(x_3, y_3)$.
- (p_1, p_2, p_3) in that order is counterclockwise if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} > 0$$

- Clockwise if the determinant is negative.
- Colinear if the determinant is zero.