APPENDIX B

ANALYSIS OF A PAIR OF RELAXATION OSCILLATORS WITH TIME DELAYS

For regions II and III in Figure 42, we show, in the singular limit, that as system (4.3) evolves, the time difference between the oscillators decreases. In region IV, we find that the time difference between the oscillators decreases only if the coupling strength is above a certain value, which we derive. Calculating how the time difference between the oscillators changes involves both 'jump-up' and 'jump-down' cases. The jump-up case is analyzed in Section B1, while the jump-down case is analyzed in Section B2. Analysis in both Appendices is for time delays $0 \le \tau < \tau_{RM}/2$.

B1 Left Branch

Region II

In Region II, O_1 jumps up, and the initial position of O_2 is such that when it receives excitation at time τ later, it jumps up directly to URB. This activity is displayed in Figure 43B. This region is bounded by $\tau < t_0 \le \tau_1 + \tau$, where we use t_0 to denote the initial time difference between the two oscillators. Assuming that O_1 jumps up at time $t = 0$, and using (4.8) and (4.9), the positions of O_1 and O_2 at time τ are given by

$$y_1(\tau) = (LLK_y - \lambda - \gamma)\, e^{-\tau} + \lambda + \gamma = c_1 e^{-\tau} + \lambda + \gamma \tag{B.1}$$

$$y_2(\tau) = (y_2(0) - \lambda + \gamma)\, e^{-\tau} + \lambda - \gamma \tag{B.2}$$

We rewrite $y_2(0)$ in terms of t_0, yielding

$$(LLK_y - \lambda + \gamma)\, e^{t_0} = c_2 e^{t_0} = (y_2(0) - \lambda + \gamma) \tag{B.3}$$

Using (B.3), we rewrite (B.2) as

$$y_2(\tau) = c_2 e^{t_0 - \tau} + \lambda - \gamma \tag{B.4}$$

To find the time difference between the oscillators after they have jumped up to the active phase, t_1, we use (4.9) to write

$$y_2(\tau) = (y_1(\tau) - \lambda - \gamma)\, e^{-t_1} + \lambda + \gamma \quad (B.5)$$

substituting (B.1) and (B.4) into (B.5) yields

$$c_2 e^{t_0 - \tau} - 2\gamma = c_1 e^{-\tau - t_1} \quad (B.6)$$

We use (B.6) to write t_1 as a function of the initial time difference, t_0,

$$t_1 = \log\left[\frac{c_1 e^{-\tau}}{c_2 e^{t_0 - \tau} - 2\gamma}\right] = g(t_0) \quad (B.7)$$

This equation arises when the order of the oscillators has been switched, i.e. O_2 leads O_1. Only in this case can the time difference between the oscillators become larger than τ. It can be shown that initial conditions in region II always map into the analogous region on the upper right branch, called region II_R. The equation for determining the time difference between the oscillators after the jump-down to the silent phase is given in (B.44). Using both (B.7) and (B.44) we can explicitly write the time difference between the oscillators after they return to the silent phase, t_2, as a function of the initial time difference, t_0,

$$t_2 = f(g(t_0)) = \log\left[\frac{c_8 e^{-\tau}(c_2 e^{t_0 - \tau} - 2\gamma)}{c_1 c_7 e^{-2\tau} + 2\gamma(c_2 e^{t_0 - \tau} - 2\gamma)}\right] \quad (B.8)$$

After some tedious algebra it can be shown that $t_0 > t_2$ for all values of $t_0 > 0$ in region II.

Region III

We now examine region III, which is defined by

$$\tau_1 + \tau < t_0 < \tau_1 + \tau_{RM} - \tau \quad (B.9)$$

where τ_{RM} is given in (4.13) and τ_1 is given in (4.14). In this region, O_1 jumps up at time $t = 0$ and O_2 receives excitation at a time τ later. O_2 is far up enough on LLB such that it hops to ULB before jumping up to URB. Typical trajectories for a pair of oscillators in this region are displayed in Figure 43C. In this case we calculate the positions of the oscillators as follows,

$$y_1(t_0 - \tau_1) = (LLK_y - \lambda - \gamma)\, e^{\tau_1 - t_0} + \lambda + \gamma = c_1 e^{\tau_1 - t_0} + \lambda + \gamma \quad (B.10)$$

$$y_2(t_0 - \tau_1) = LLK_y + \alpha_R \quad (B.11)$$

To find the time difference between the oscillators on the active phase, t_1, we use (4.9) to write

$$c_1 e^{\tau_1 - t_0} = (LLK_y + \alpha_R - \lambda - \gamma)\, e^{-t_1} = c_5 e^{-t_1} \quad (B.12)$$

We use (B.12) to write t_1 as a function of the initial time difference, t_0,

$$t_1 = t_0 - \tau_1 - \tau_5 \tag{B.13}$$

where

$$\tau_5 = \log\left(\frac{c_1}{c_5}\right) \tag{B.14}$$

In this case it can be shown that the values of t_1 can be in both regions II_R and III_R. We do not examine region II_R as these time differences can be shown to always map to region II, and then the time difference between the oscillators always decreases as discussed previously. When now examine the time difference between the oscillators when they map from region II to region III_R. Using (B.13) and (B.49) we write the time difference between the oscillators after they have both returned to the silent phase, t_2, as

$$t_2 = t_0 - \tau_1 - \tau_5 - \tau_2 - \tau_4 \tag{B.15}$$

The values of τ_2 and τ_4 arise from the analysis of the jump-down and are given in (B.41) and (B.50) respectively. We examine what values of t_0 result in a value of t_1 that is in region III_R, or

$$t_1 > \tau_2 + \tau \tag{B.16}$$

$$t_0 - \tau_1 - \tau_5 > \tau_2 + \tau \tag{B.17}$$

$$t_0 > \tau_1 + \tau_5 + \tau_2 + \tau \tag{B.18}$$

If t_2 is positive, then (B.15) shows that the time difference between the oscillators after a single period has decreased. However, if t_2 is negative, we must test if it is possible for $|t_2| > t_0$. We examine the following inequality,

$$t_2 = t_0 - \tau_1 - \tau_5 - \tau_2 - \tau_4 > -t_0 \tag{B.19}$$

$$2t_0 > \tau_1 + \tau_5 + \tau_2 + \tau_4 \tag{B.20}$$

Using (B.20) and the minimum value of t_0, from (B.18), results in

$$2(\tau_1 + \tau_5 + \tau_2 + \tau) > \tau_1 + \tau_5 + \tau_2 + \tau_4 \tag{B.21}$$

$$\tau_1 + \tau_5 + \tau_2 + 2\tau > \tau_4 \tag{B.22}$$

Since $\tau_1 > \tau_4$, (B.22) is always true, and therefore $|t_2| < t_0$. Thus, the oscillators in region III always experience a decrease in the absolute value of their time difference after jumping up and down.

Region IV

We now examine region IV. The last case we examine on LLB is for initial conditions such that O_1 jumps up to URB and down to LLB, while O_2 remains on the silent phase (see Figure 43D). This region is defined by the following bounds,

$$\tau_1 + \tau_{RM} + \tau < t_0 \le \tau_{LLB} - \tau \tag{B.23}$$

where τ_{LLB} is given in (4.11). The leading oscillator, O_1, jumps up at time $t = 0$, and the positions of the oscillators after O_1 traverses LRB are

$$y_1(\tau_{RM}) = LRK_y \tag{B.24}$$

$$y_2(\tau_{RM}) = c_2 e^{t_0 - \tau_{RM}} + \lambda - \gamma \tag{B.25}$$

To find the time difference between the oscillators when they are both on the silent phase, t_2, we use (4.8) to write

$$LRK_y = c_2 e^{t_0 - \tau_{RM} - t_2} + \lambda - \gamma \tag{B.26}$$

We use (B.26) to write t_2 as a function of t_0,

$$t_2 = t_0 - \tau_6 - \tau_{RM} \tag{B.27}$$

with

$$\tau_6 = \log\left(\frac{c_4}{c_2}\right) \tag{B.28}$$

If t_2 is positive, then a decrease in the time difference between the oscillators occurs. However, if t_2 is negative, then O_2 leads O_1, and we must check if $|t_2| > t_0$. We examine the following equation

$$\tau_6 + \tau_{RM} - t_0 < t_0 \tag{B.29}$$

$$t_0 > \frac{\tau_6 + \tau_{RM}}{2} \tag{B.30}$$

Using the minimum value of t_0 in region IV, the constraint (B.30) becomes

$$2\,(\tau_1 + \tau_{RM} + \tau) > \tau_6 + \tau_{RM} \tag{B.31}$$

or,

$$\tau_6 < 2\tau_1 + \tau_{RM} + 2\tau \qquad \text{(B.32)}$$

$$\frac{c_4}{c_2} < \left(\frac{c_6}{c_2}\right)^2 \frac{c_1}{c_3} e^{2\tau} \qquad \text{(B.33)}$$

$$c_6^2 \;=\; (c_2 + \alpha_R)^2 > \frac{c_2 c_3 c_4}{c_1} e^{-2\tau} \qquad \text{(B.34)}$$

We rewrite (B.34) as a restriction on the coupling strength,

$$\alpha_R > \sqrt{\frac{c_2 c_3 c_4}{c_1}} e^{-\tau} - c_2 \qquad \text{(B.35)}$$

The above condition puts a limiting value on the minimum value of α_R. Below this value for the connection weight it is possible for $|t_2| > t_0$ and neutrally stable desynchronous solutions can result.

There is a part of region V that juts between region IV and region III. In region V, it is possible for antiphase solutions to arise dependent on the initial conditions, the time delay, and the coupling strength. We examine whether a pair of oscillators with initial conditions in region IV will automatically map to region III, without entering region V. To test this we make the following statement,

$$t_2 \;=\; \tau_6 + \tau_{RM} - t_0 < \tau_1 + \tau_{RM} - \tau \qquad \text{(B.36)}$$

$$\tau_6 - \tau_1 + \tau < t_0 \qquad \text{(B.37)}$$

Using (B.37) with the smallest value that t_0 can take in region IV, we write

$$\tau_6 - \tau_1 + \tau < \tau_1 + \tau_{RM} + \tau \qquad \text{(B.38)}$$

$$\tau_6 - \tau_{RM} < 2\tau_1 \qquad \text{(B.39)}$$

$$\alpha_R > \sqrt{\frac{c_4 c_3}{c_1}} - c_2 \qquad \text{(B.40)}$$

If condition (B.40) is satisfied then the oscillators whose initial conditions are in region IV will not map into region V. We note that if condition (B.35) is satisfied, then automatically, (B.40) is satisfied. Thus for coupling strengths that result in loose synchrony, a pair of oscillators in region IV will never map to region V. Also, we note that it is not possible for O_1 to make two traversals of LRB because the maximum possible y-value for O_2 is $LRK_y + \alpha_R$.

If constraints (B.35) and the constraint derived in Appendix B, (B.44), are satisfied then the oscillators whose initial time difference is in regions II-IV of Figure 42, will eventually have a time difference of less than or equal to the time delay, i.e. they will be loosely synchronous.

B2 Right Branch

We calculate how the time difference between the oscillators changes as they jump down from the active phase to the silent phase. We label the two regions II_R and III_R because of their correspondence to regions II and III of Figure 42. No other trajectories are possible since we have assumed that the fastest branch of the system is LRB.

We assume that O_1 jumps down first, at time $t = 0$, then O_2 ceases to receive excitation at time τ and jumps down to LLB (Figure 43E). The time difference between the oscillators in region II_R is bounded by $\tau < t_1 \le \tau_2 + \tau$, where

$$\tau_2 = \log\left(\frac{c_3}{c_7}\right) \tag{B.41}$$

The positions of the oscillators immediately after the jump-down are given by

$$y_1(\tau) = c_8 e^{-\tau} + \lambda - \gamma \tag{B.42}$$

$$y_2(\tau) = c_7 e^{t_1 - \tau} + \lambda + \gamma \tag{B.43}$$

The time difference between the oscillators after they are both on the silent phase, t_2, is given by

$$t_2 = \log\left[\frac{c_8 e^{-\tau}}{c_7 e^{t_1 - \tau} + 2\gamma}\right] = f(t_1) \tag{B.44}$$

It can be shown that the oscillators whose initial conditions are in region II_R always map to region II of Figure 42.

Region III_R

The second case to examine is the region

$$\tau_2 + \tau < t_1 \le \tau_2 + \tau_{RM} - \tau \tag{B.45}$$

where τ_{RM} is defined in (4.13). In this region, O_1 jumps down, and the initial separation between the oscillators is such that O_2 hops to LRB before jumping down to LLB. Trajectories for a pair of oscillators in this region are shown in Figure 43F. The positions of the oscillators when they are both on LLB are given by

$$y_1(t_1 - \tau_2) = (LRK_y + \alpha_R - \lambda + \gamma)\, e^{\tau_2 - t_1} + \lambda - \gamma = c_8 e^{\tau_2 - t_1} + \lambda - \gamma \qquad \text{(B.46)}$$

$$y_2(t_1 - \tau_2) = LRK_y \qquad \text{(B.47)}$$

To find the time difference between the oscillators when they are both on the silent phase, t_2, we use (4.8) to write

$$c_8 e^{\tau_2 - t_1} = c_4 e^{-t_2} \qquad \text{(B.48)}$$

resulting in

$$t_2 = t_1 - \tau_2 - \tau_4 \qquad \text{(B.49)}$$

where

$$\tau_4 = \log\left(\frac{c_8}{c_4}\right) \qquad \text{(B.50)}$$

Other trajectories can arise during the jump-down, but not with the assumptions that LRB is the fastest branch in the system. This assumption implies that the following condition must be true,

$$\tau_{RM} < \tau_9 \qquad \text{(B.51)}$$

where

$$\tau_9 = \log\left(\frac{c_8}{c_6}\right) \qquad \text{(B.52)}$$

From condition (B.51) we derive the following bound on the coupling strength,

$$\frac{c_1}{c_3} < \frac{c_4 + \alpha_R}{c_2 + \alpha_R} \qquad \text{(B.53)}$$

$$\alpha_R < \frac{c_1 c_2 - c_3 c_4}{c_3 - c_1} \qquad \text{(B.54)}$$