Parameter-free Topology Inference and Sparsification of Data on Manifolds

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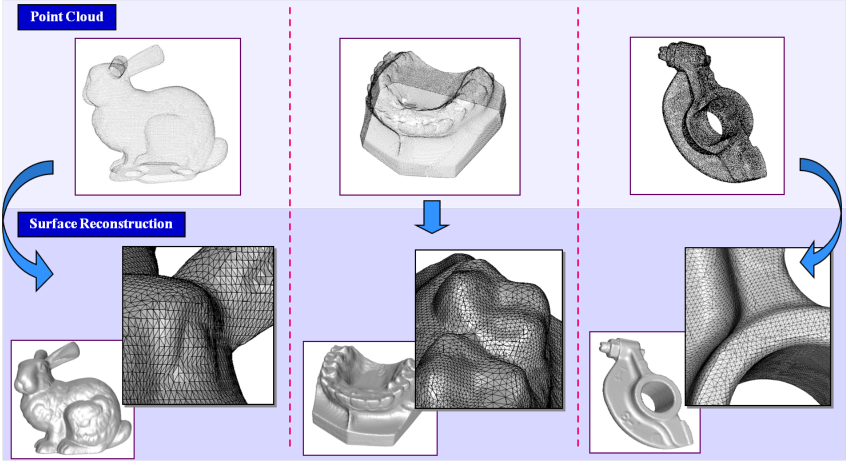
Joint work with Zhe Dong and Yusu Wang

Surface Reconstruction



- Crust, Cocone are parameter-free

Point cloud→Complex→Homology inference

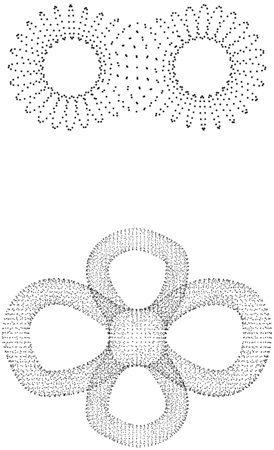
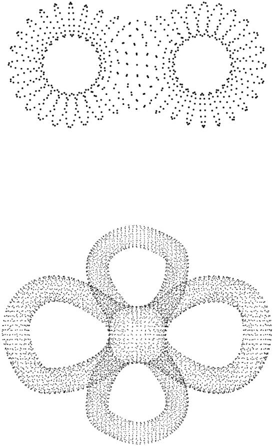


Figure: Point cloud

Point cloud→Complex→Homology inference



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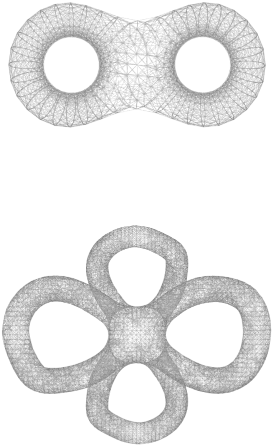

Figure: Rips complex

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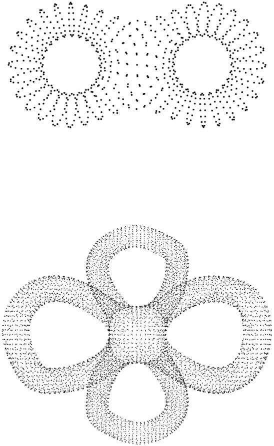

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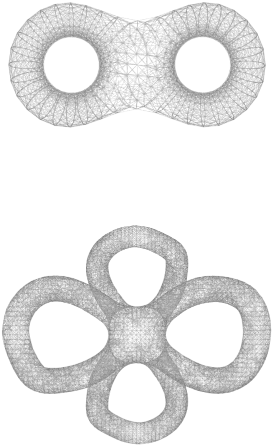

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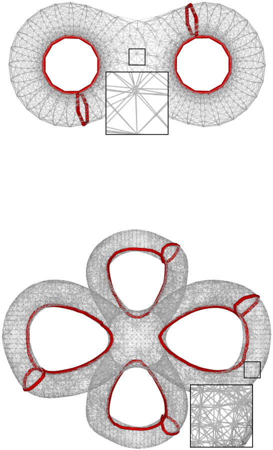
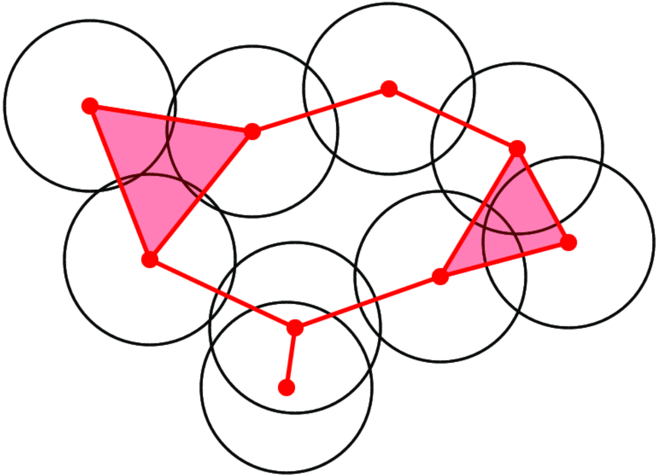


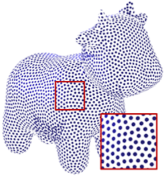
Figure: Loops

Rips complex $\mathcal{R}^\alpha(P)$: parameter α



Sample density

- Globally uniform sample
  - Estimate α by NN distance
  - Works, but stringent requirement
  - Assume dense wrt smallest feature size ρ and assume parameter $\alpha << \rho$



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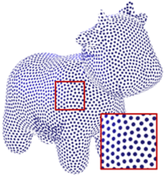
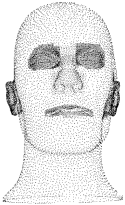
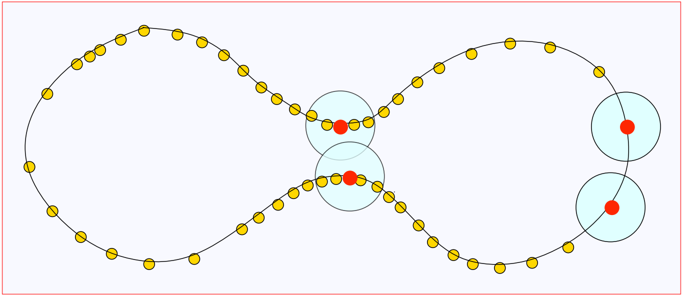
- Locally dense sample
  - Sample density varies with "local" feature size
  - How to estimate α
  - Worse, there may not exist a 'global' α;

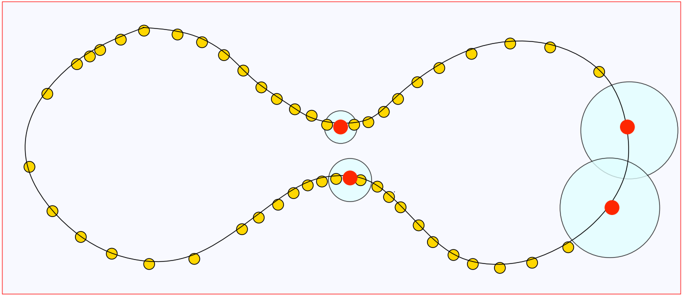




Difficulty with global α for locally dense sampling



Adaptive α



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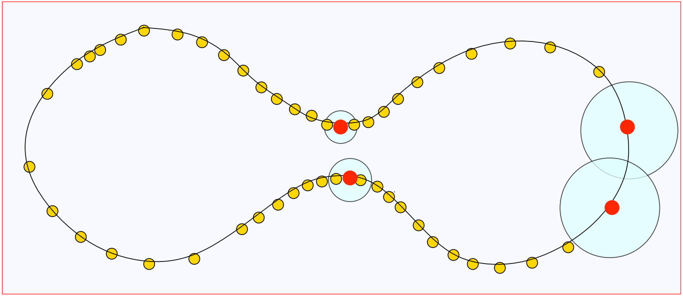
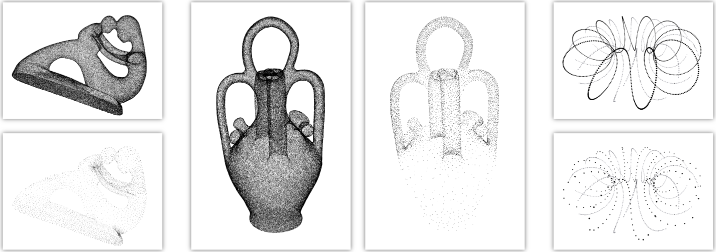


Goal:

- compute a function $f : P \rightarrow \mathbb{R}$ bounded by 'lfs' and 'wfs'
- adjust sample density following $f \implies$ sparsification

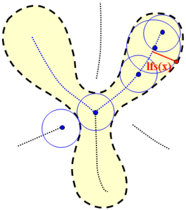
Data Sparsification

- $P \subset \mathcal{M} \subset \mathbb{R}^d$, a dense discrete sample of a manifold.
- Compute $Q \subset P$ so that
  - ▷ $|Q| << |P|$ and 'locally uniform'
  - ▷ topology of $\mathcal{M}$ can still be inferred from Q



Local feature size

- Medial axis $A := A(\mathcal{M})$



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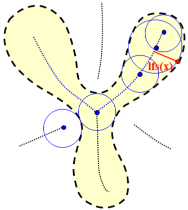

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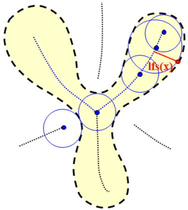
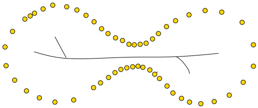


- Local feature size $lfs(x) := d(x, A)$;
- Locally ε-dense sample P [ABE 1998]
  - $\forall x \in \mathcal{M}$, $\exists p \in P$ such that $d(x,p) \leq \varepsilon \cdot lfs(x)$;



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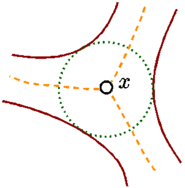

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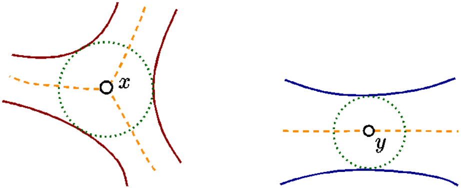

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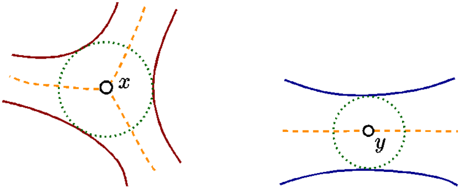
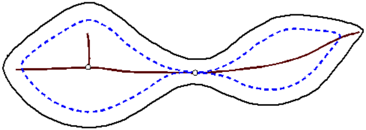


- Offset topology changes at critical points



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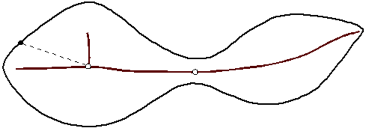

Locally uniform sample

Definition

Q is δ-sparse ε-dense sample wrt $f : \mathcal{M} \to \mathbb{R}$ if

- $\forall x \in \mathcal{M}$, $\exists q \in Q$, $d(x, q) \leq \varepsilon f(x)$
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  - which can be computed efficiently
  - also satisfies $\forall x \in \mathcal{M}$, $c_1 \cdot lfs(x) \leq Lnfs(x) \leq c_2 \cdot lwfs(x)$

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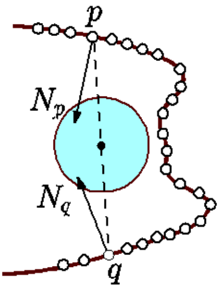

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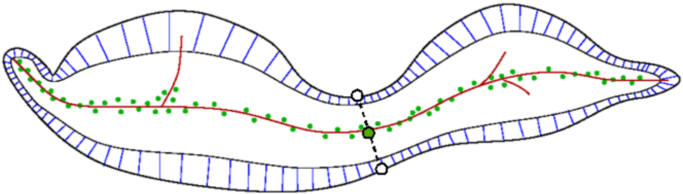

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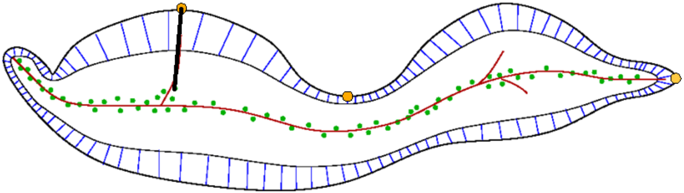

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Theorem (Approximation)

Given a local ε-dense sample of $\mathcal{M}$, for any $x \in \mathcal{M}$, one has:

$$c_1 \cdot lfs(x) \leq Lnfs_\beta(x) \leq c_2 \cdot lwfs(x)$$

- c_1, c_2 *depend on* β *and* ε
- *In particular,* β *close to* $\frac{\pi}{4}$ *works for* ε *small enough*

Sparsification using Lnfs

- L_β: computed lean set; $Lnfs_\beta(p) := d(p, L_\beta)$
- δ: sparsification constant (can be chosen to depend only on β)
- $\textsc{Lean}(P,\ \beta,\delta)$ by iterative deletions
  - Put each $p \in P$ in max priority queue with priority $Lnfs_\beta(p)$
  - While (queue not empty)
    - ⋆ $q \leftarrow Extractmax(queue)$;
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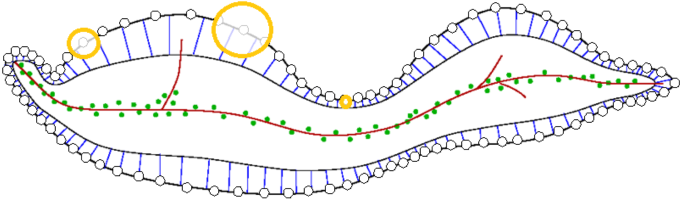

Sparsification

Theorem (Sparsification)

Given a local ε*-dense sample* $P \subset \mathcal{M}$*, the* $\textsc{Lean}(P,\beta,\delta)$ *computes a* δ*-sparse,* $\frac{4}{3}\delta$*-dense sample* $Q \subset P$ *wrt* $Lnfs_\beta(\cdot)$.

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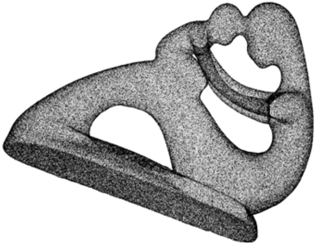
Given a local ε-dense sample $P \subset \mathcal{M}$, the $\textsc{Lean}(P,\beta,\delta)$ *computes a δ-sparse, $\frac{4}{3}\delta$-dense sample $Q \subset P$ wrt $Lnfs_\beta(\cdot)$.*



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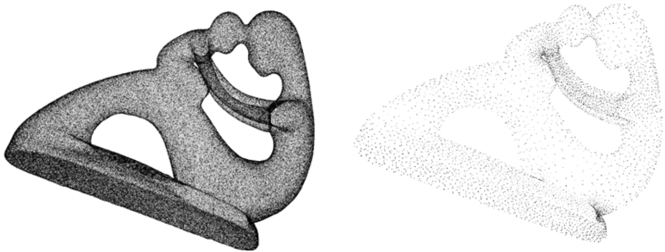

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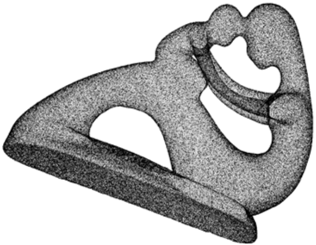
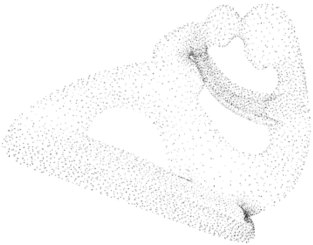


126500 points



6016 points

Homology inference by scaled distance

- Define scaled distance $h(x) := \frac{d(x,\mathcal{M})}{d(x,\mathcal{M})+Lnfs_{\beta}(x)}$
- Offset wrt h: $\mathcal{M}_{\alpha} := h^{-1}[0,\alpha]$

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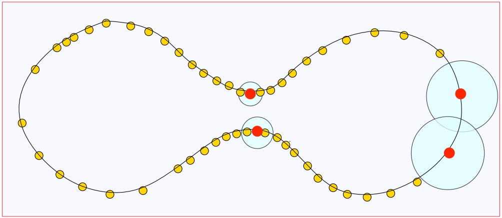

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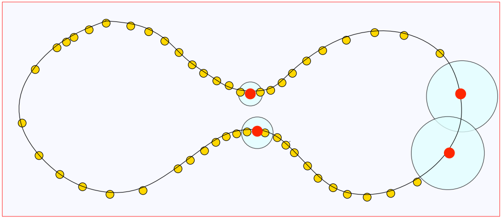

Proposition

If P is a δ-dense wrt $Lnfs_\beta(\cdot)$, then for $\alpha > 0$

$$\mathcal{M}_{\frac{\alpha}{1+2\alpha}} \subseteq P_{\alpha+\delta+\alpha\delta} \subseteq \mathcal{M}_{\alpha+\delta+\alpha\delta}$$

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For $\beta < \theta < \frac{\pi}{4}$, $\alpha + \delta \leq \frac{1}{3}\frac{\cos 2\theta}{1+\cos 2\theta}$, *and* P *a* δ*-dense sample wrt* $Lnfs_{\beta}(\cdot)$, *one has:*

$$\mathrm{i}m\left(H_i(C^{\alpha+\delta}(P)) \rightarrow H_i(C^{3(\alpha+\delta)}(P))\right) \cong H_i(\mathcal{M})$$

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For $\beta < \theta < \frac{\pi}{4}$, $\alpha + \delta \leq \frac{1}{3}\frac{\cos 2\theta}{1+\cos 2\theta}$, and P a δ-dense sample wrt $Lnfs_\beta(\cdot)$, one has:

$$\mathrm{im}\left(H_i(C^{\alpha+\delta}(P)) \to H_i(C^{3(\alpha+\delta)}(P))\right) \cong H_i(\mathcal{M})$$

- Adaptive Rips
$R^\alpha(P) := \{\sigma \mid d(p,q) \leq \alpha \cdot \left(Lnfs_\beta(p) + Lnfs_\beta(q)\right)\}$
- Interleaving: $C^\alpha(P) \subseteq R^\alpha(P) \subseteq C^{2\alpha}(P)$

Homology inference

Theorem

- $\beta = \frac{\pi}{5}$, $\delta = \frac{1}{26} \frac{\cos 2\beta}{1+\cos 2\beta}$
- *$Q \subseteq P$ be δ-sparse, $\frac{4}{3}\delta$-dense wrt $Lnfs_\beta(\cdot)$ where $P \subset \mathcal{M}$ is locally ε-dense*

Then, $\mathrm{im}\left(H_i(R^{2\delta}(Q)) \to H_i(R^{12\delta}(Q))\right) \cong H_i(\mathcal{M})$

Homology inference

Theorem

- $\beta = \frac{\pi}{5}$, $\delta = \frac{1}{26}\frac{\cos 2\beta}{1+\cos 2\beta}$
- *$Q \subseteq P$ be δ-sparse, $\frac{4}{3}\delta$-dense wrt $Lnfs_\beta(\cdot)$ where $P \subset \mathcal{M}$ is locally ε-dense*

Then, $\mathrm{im}\left(H_i(R^{2\delta}(Q)) \to H_i(R^{12\delta}(Q))\right) \cong H_i(\mathcal{M})$

- Compute Q by $\textsc{Lean}(P, \beta, \delta)$
- Compute persistence $H_i(R^{2\delta}(Q)) \to H_i(R^{12\delta}(Q))$ where $R^\alpha(P) := \{\sigma \mid d(p,q) \leq \alpha \cdot \left(Lnfs_\beta(p) + Lnfs_\beta(q)\right)\}$
- Each $q \in Q$ has $O(1)$ neighbors, so complex size linear in $|Q|$

Open questions

- Extending lean-set based sparsification to noisy data?

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- What about non-manifolds?
- Potential use in topological data analysis.

Thank you !