

Convergence, Stability, and Discrete Approximation of Laplace Spectra*

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Abstract

Spectral methods have been widely used in a broad range of applications fields. One important object involved in such methods is the Laplace-Beltrami operator of a manifold. Indeed, a variety of work in graphics and geometric optimization uses the eigen-structures (i.e. the eigenvalues and eigenfunctions) of the Laplace operator. Applications include mesh smoothing, compression, editing, shape segmentation, matching, parameterization, and so on. While the Laplace operator is defined (mathematically) for a smooth domain, these applications often approximate a smooth manifold by a discrete mesh. The spectral structure of the manifold Laplacian is estimated from some discrete Laplace operator constructed from this mesh.

In this paper, we study the important question of how well the spectrum computed from the discrete mesh approximates the true spectrum of the manifold Laplacian. We exploit a recent result on mesh Laplacian and provide the first convergence result to relate the spectrum constructed from a general mesh (approximating an m -manifold embedded in \mathbb{R}^d) with the true spectrum. We also study how stable these eigenvalues and their discrete approximations are when the underlying manifold is perturbed, and provide explicit bounds for the Laplacian spectra of two “close” manifolds, as well as a convergence result for their discrete approximations. Finally, we present various experimental results to demonstrate that these discrete spectra are both accurate and robust in practice.

1 Introduction

Spectral methods have been used in a broad range of applications fields, including computer vision, machine learning and data mining. An important object involved in such methods is the Laplace-Beltrami operator of a given manifold (such as a surface). It is a fundamental object encoding the intrinsic geometry of the underlying manifold, and has many properties useful for practical applications. For example, eigenfunctions of the Laplacian form a natural basis for square integrable functions on the manifold analogous to Fourier harmonics for functions on a circle (i.e. periodic functions). Such a basis reflects the intrinsic geometry of the manifold, which has been used to perform various tasks like dimensionality reduction, motion tracking, and surface matching. Its relation to the heat diffusion also makes it a primary tool for surface smoothing. Indeed, in recent

years, a considerable amount of work in graphics and geometric optimization use the eigen-structures (i.e. the eigenvalues and eigenfunctions) of the Laplace operator, and applications include mesh smoothing, compression, editing, shape segmentation, matching, and parameterization (see reviews [7, 14, 24]).

While the Laplace operator is defined (mathematically) for a smooth domain, in various applications, the input object is typically represented by a (discrete) mesh that approximates the underlying smooth object. Hence in practice, its spectral structure is estimated from some discrete analog of the Laplace operator constructed from the input mesh. An important question is whether this discrete approximation of the Laplacian eigen-structure is accurate or not. This is the first question we aim to address in this paper. We further study in this paper how stable these eigen-structures and their discrete approximations are when the underlying manifold is perturbed, as robustness is usually an important property for practical applications such as in shape matching.

Related work and new results. Several discretizations of the Laplace operator for meshes have been proposed. See [21] for a nice discussion explaining the diversity of discrete Laplace operators. One of the most popular ones is the so-called *cotangent scheme* for surfaces embedded in three-dimensional space, originally proposed in [5, 12], and its variants [4, 10, 11, 22]. The cotangent scheme has several nice properties, including the so-called *weak convergence* (which, roughly speaking, means convergence in the sense of inner product) [6, 20]. However, in general, it does not provide the standard pointwise convergence [22, 23], though there are some convergence results for certain special meshes and manifolds [22]. Nevertheless, in his Ph.D dissertation, Wardetzky showed a convergence result for spectra based on the cotangent scheme when the surface mesh satisfies some mild conditions on the aspect ratio of the triangles [19]. Reuter et al. computed a discrete Laplace operator using the finite element method, and obtained good practical performance [13].

In [3], Belkin et al. proposed the so-called mesh-Laplace operator, which is the first discrete Laplacian that *pointwise converges* to the true Laplacian as the

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input mesh approximates a smooth manifold better. Specifically, for any C^2 -smooth scalar function f defined on a manifold M and its restriction \hat{f} on vertices of a mesh K , $|\Delta_M f(x) - \mathbf{D}_K \hat{f}(x)|_\infty$ converges to zero as K converges to M , where Δ_M and \mathbf{D}_K denote the Laplacian of M and its discrete approximation from K , respectively. This result can be easily extended to higher dimensional manifolds¹. Experimental results also show that this operator indeed produces accurate approximation of the Laplace operator under various conditions, such as noisy data input, and different sampling conditions etc [16].

However, so far, no general convergence result is known for the eigen-structures of any discrete Laplacian for meshes in arbitrary dimensions, even though many practical applications rely on these structures. In general, pointwise convergence between two operators is not strong enough to imply the convergence of their respective eigenvalues nor eigenfunctions. As mentioned above, partial spectrum convergence result was obtained for surface meshes based on the cotangent scheme [19]. For high dimensional manifolds, convergence result is known only under the statistical setting — if input points are *randomly* sampled from the underlying manifold, Belkin and Niyogi showed that the eigen-structure of the weighted graph Laplacian of these points converges to that of the manifold Laplacian [2].

In Section 4, we present the first result relating the eigen-structure of some discrete Laplacian from meshes with the manifold Laplacian for m -manifolds embedded in \mathbb{R}^d . We focus on the mesh-Laplacian proposed in [3] and show that its eigenvalues converge to those of the manifold Laplacian as the mesh approximates a smooth manifold better. The new result is achieved by showing that the mesh-Laplace operator converges to the manifold Laplacian not only pointwise, but in fact under a stronger operator norm when considered in a certain appropriate Sobolev space.

In Section 3, we investigate a related question of how stable the Laplacian spectrum and its discrete approximation are as the underlying manifold is perturbed. We give explicit bounds for the Laplacian spectra of two “close by” manifolds, and present a convergence result for their discrete approximations. This is the first stability result for discrete Laplace operators.

In Section 5, we provide experimental evidence showing that the mesh Laplacian indeed produces good estimates of spectra of the manifold Laplacian, and is robust to noise and deformations.

¹The extension to d -manifolds embedded in \mathbb{R}^{d+1} is straightforward. When the co-dimension is greater than 1, one needs to define the sampling condition appropriately to guarantee the convergence of the normal space.

2 Approach Overview

2.1 Objects and Notations

Laplace-Beltrami operator. Consider a smooth, compact manifold M of dimension m isometrically embedded in some Euclidean space \mathbb{R}^d . The *medial axis* of M is the closure of the set of points in \mathbb{R}^d that have at least two closest points in M . The *reach* $\rho(M)$ of M is the infimum of the closest distance from any point $p \in M$ to the medial axis of M . In this paper, we assume that the manifold M has a positive reach.

Given a twice continuously differentiable function $f \in C^2(M)$, let $\nabla_M f$ denote the gradient vector field of f on M . The *Laplace-Beltrami operator* Δ_M of f is defined as the divergence of the gradient; that is, $\Delta_M f = \text{div}(\nabla_M f)$. For example, if M is \mathbb{R}^2 , then its Laplacian has the familiar form $\Delta_{\mathbb{R}^2} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

Discrete setting. In practice, the underlying manifold is often approximated by a discrete mesh. Given a simplicial mesh K with all vertices lying on M , we say that it ε -*approximates* a smooth manifold M if (i) for any point $p \in M$, there is a sample point (i.e, a vertex) from K that is at most $\varepsilon\rho(M)$ away; and (ii) the projection map ϕ from the underlying space $|K|$ of K onto M is a homeomorphism and its Jacobian is bounded by $1 + O(\varepsilon)$ at any point in the interior of the m -simplices. Intuitively, the first condition ensures that the mesh is sufficiently fine. However, a very fine mesh can still provide a poor approximation to the underlying surface. Hence we need the second condition to ensure that the distortion between $|K|$ and M is small. We remark that for an m -manifold embedded in \mathbb{R}^{m+1} (such as a surface embedded in \mathbb{R}^3), such an ε -approximation is equivalent to the (ε, η) -approximation used in [3] with $\eta = O(\varepsilon)$, which bounds both the sampling density and the normal deviation.

In the discrete setting, an input function f is only available at vertices of K , and thus can be represented as an n -dimensional vector $\hat{\mathbf{f}} = [f(v_1), \dots, f(v_n)]^T$ where $V = \{v_1, \dots, v_n\}$ is the set of vertices in K . In [3], a discrete mesh-Laplacian \mathbf{L}_t^K was proposed, where t is some parameter. Being a linear operator, this discrete analog of the Laplace operator is an n by n matrix. It is defined by:

$$\mathbf{L}_t^K f(v_i) = \frac{1}{t(4\pi t)^{m/2}} \sum_{v_j \in V} A_j e^{-\frac{\|v_i - v_j\|^2}{4t}} (f(v_i) - f(v_j)),$$

where A_j is $\frac{1}{m+1}$ -th of the total volume of all m -simplices incident to the vertex v_j . This discrete operator \mathbf{L}_t^K pointwise converges to the Laplace operator Δ_M of M . More precisely,

THEOREM 2.1. ([3]) *Set $t(\varepsilon) = \varepsilon^{\frac{1}{2.5+\alpha}}$ for an arbitrary fixed positive number $\alpha > 0$. Then for any $f \in C^2(M)$ and any point $x \in M$,*

$$\lim_{\varepsilon \rightarrow 0} \sup_{K(\varepsilon)} |\mathbf{L}_{t(\varepsilon)}^{K(\varepsilon)} \mathbf{f}(x) - \Delta_M f(x)| = 0,$$

where the supremum is taken over all ε -approximations $K(\varepsilon)$ of M .

Problem definition. In this paper, we aim to understand the stability of the spectrum of the Laplace operator and its discrete analog. The first question we consider is:

- P1. How does the spectrum of the mesh-Laplacian \mathbf{L}_t^K relate to that of Δ_M . Does the former converge to the latter as the sampling becomes denser?

The second problem aims to understand the stability of the Laplacian spectrum (both the continuous and discrete versions) when the underlying manifold M is perturbed. Specifically, given two smooth and compact m -manifolds M and N embedded in \mathbb{R}^d , we say that M and N are δ -close if there is a homeomorphism $\Psi : M \rightarrow N$ such that (1) $\|x - \Psi(x)\| = O(\delta)$ for any $x \in M$, and (2) the Jacobian of the map Ψ is bounded by $|\mathcal{J}\Psi - 1| = O(\delta)$ at any point of M .

- P2. How are the spectra of Δ_M and Δ_N , as well as the spectra of the discrete Laplacian computed from meshes approximating M and N , related.

2.2 Overview of Approaches and Results

To connect the Laplace operator and its approximation, we need an intermediate operator \mathcal{L}_t^M , called the *functional approximation* of Δ_M , first introduced in [1]. Given a point $p \in M$ and a function $f : M \rightarrow \mathbb{R}$, it is defined as:

$$(2.1) \quad \mathcal{L}_t^M f(x) = \frac{1}{t(4\pi t)^{m/2}} \int_{y \in M} e^{-\frac{\|x-y\|^2}{4t}} (f(x) - f(y)) dy.$$

The intuition behind using this operator is two-fold. First, the closed form of the Laplace operator is unavailable for general manifolds, making it hard to analyze directly. Secondly, while the Laplace operator is an unbounded operator, this functional Laplacian is bounded with a simple spectral structure. This facilitates us to use the standard perturbation theory to analyze the stability of this operator. The connection between the functional Laplacian and Δ_M can be summarized in the following theorem [1, 2].

THEOREM 2.2. ([1, 2]) *For a function $f \in C^2(M)$, we have that*

$$\lim_{t \rightarrow 0} \|\mathcal{L}_t^M f - \Delta_M f\|_\infty = 0.$$

Furthermore, let $\{\lambda_i\}$ and $\{\hat{\lambda}_i\}$ denote the discrete eigenvalues of Δ_M and \mathcal{L}_t^M enumerated in non-decreasing order. Then, for any fixed i and for t small enough (more precisely, $t < \frac{1}{2\lambda_i}$), we have $|\lambda_i - \hat{\lambda}_i| = O(t^{\frac{2}{m+6}})$.

In [3], it was shown that given a mesh K that ε -approximates M , \mathcal{L}_t^M can be approximated by the mesh Laplacian \mathbf{L}_t^K with pointwise convergence guarantee. When combined with the above theorem, this implies Theorem 2.1. However, to answer Question P1, we need a stronger (than pointwise) convergence result between \mathbf{L}_t^K and \mathcal{L}_t^M . Specifically, in Section 4, we show the following result, which is obtained by bounding the operator norm of the difference between \mathbf{L}_t^K and \mathcal{L}_t^M in an appropriate functional space.

THEOREM 2.3. *Given a smooth m -manifold M , let $K(\varepsilon)$ denote a simplicial mesh K that ε -approximates M . Let $\{\hat{\lambda}_i\}$ and $\{\lambda_i^D(\varepsilon)\}$ denote the set of non-decreasing discrete eigenvalues of \mathcal{L}_t^M and of $\mathbf{L}_t^{K(\varepsilon)}$, respectively. Then, for any fixed i , we have that $\lim_{\varepsilon \rightarrow 0} |\hat{\lambda}_i - \lambda_i^D(\varepsilon)| = 0$.*

This result, combined with Theorem 2.2, gives an answer to Question P1 of this paper, which is stated below. The relation between these results is illustrated in Figure 1.

THEOREM 2.4. *Given a smooth m -manifold M and a simplicial mesh $K(\varepsilon)$ that ε -approximates M , let $\{\lambda_i\}$ and $\{\lambda_i^D(\varepsilon)\}$ denote the set of non-decreasing discrete eigenvalues of Δ_M and of $\mathbf{L}_t^{K(\varepsilon)}$, respectively. Then, for any fixed i , we have that $\lim_{t, \varepsilon, \frac{\varepsilon}{t^{\frac{m}{2}+3}} \rightarrow 0} |\lambda_i - \lambda_i^D(\varepsilon)| = 0$.*

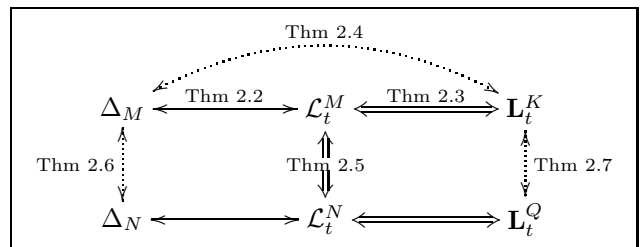


Figure 1: Theorems relating different operators are shown on top of the arrows. Double arrows indicate the two main new results in this paper, and lead to those results specified by dotted arrows.

To answer Question P2, the main component is a perturbation result for the functional Laplace operator. Specifically, let $\text{Spec}(A)$ denote the spectrum of an operator A . We show that:

THEOREM 2.5. *Given two δ -close m -manifolds M and N , the Hausdorff distance between $\text{Spec}(\mathcal{L}_t^M)$ and $\text{Spec}(\mathcal{L}_t^N)$ is $O(\frac{\delta}{t^{\frac{m}{4}+2}})$. That is, for any eigenvalue $\hat{\lambda} \in \text{Spec}(\mathcal{L}_t^M)$ and $\hat{\omega} \in \text{Spec}(\mathcal{L}_t^N)$, we have that $\text{dist}(\hat{\lambda}, \text{Spec}(\mathcal{L}_t^N)) = O(\frac{\delta}{t^{\frac{m}{4}+2}})$ and $\text{dist}(\hat{\omega}, \text{Spec}(\mathcal{L}_t^M)) = O(\frac{\delta}{t^{\frac{m}{4}+2}})$, where $\text{dist}(x, X) := \inf_{y \in X} |y - x|$.*

Combining this result with Theorem 2.2 bounds the spectra of Δ_M and of Δ_N (Theorem 2.6 below); and combining it with Theorem 2.3 leads to spectral convergence of discrete Laplacians for meshes approximating M and N , as N converges to M (Theorem 2.7 below). These relations are also illustrated in Figure 1.

THEOREM 2.6. *Let $\{\lambda_i\}$ and $\{\omega_i\}$ be the nondecreasing eigenvalues of Δ_M and Δ_N with multiplicity. Then, for any λ_i , there exists $\delta_0 > 0$ such that if M and N are δ -close for any $\delta < \delta_0$, then $|\lambda_i - \omega_i| = O(\delta^{\frac{8}{m^2+14m+56}})$.*

THEOREM 2.7. *Let M and N be two m -manifolds that are δ -close, and $K(\varepsilon)$ and $Q(\varepsilon)$ be two simplicial meshes ε -approximating M and N , respectively. Let $\{\lambda_i^D\}$ and $\{\omega_i^D\}$ be the non-decreasing eigenvalues of \mathbf{L}_t^K and \mathbf{L}_t^Q with multiplicity. Then, for any fixed i , we have that as N converges to M and as the meshes approximate better, $\lim_{\delta, \varepsilon, \frac{\varepsilon}{\delta^{\frac{m}{2}+3}}, \frac{\delta}{t^{\frac{m}{4}+2}} \rightarrow 0} |\lambda_i^D - \omega_i^D| = 0$.*

Outline. In the rest of this paper, instead of following the above order where we introduced the results, we first prove Theorem 2.5 and 2.6 in Section 3, as this will illustrate some of the main ideas of our approach. The proof for Theorem 2.3 is more technical, and we will present a sketch of it as well as proofs for the remaining results in Section 4.

3 Perturbation of Manifold and Stability

In this section, we study the behavior of the spectrum of Δ_M and its discrete approximation as the underlying manifold M is perturbed to another manifold N that is δ -close to M . The main component is to relate the spectrum of \mathcal{L}_t^M with that of \mathcal{L}_t^N (i.e., Theorem 2.5) which we focus on now. Here we consider the Hilbert spaces $L^2(M)$ and $L^2(N)$, which are the spaces of square integrable functions on M and on N , respectively. Notice that for any compact manifold X , the functional Laplacian \mathcal{L}_t^X is a self-adjoint and bounded operator in $L^2(X)$ (equipped with the standard L_2 norm).

Roughly speaking, if the norm of the difference between two operators is bounded in some space, then the distance between their spectra is also bounded. Hence, we wish to bound the operator norm of $\mathcal{L}_t^M - \mathcal{L}_t^N$. However, the two operators \mathcal{L}_t^M and \mathcal{L}_t^N are defined over two different spaces, $L^2(M)$ and $L^2(N)$, respectively. Thus, they are not directly comparable. Now assume $\Psi : M \rightarrow N$ is a homeomorphism between M and N that satisfies the δ -closeness conditions. We compare the operator \mathcal{L}_t^M with the pull-back operator of \mathcal{L}_t^N . Specifically, given an operator $\mathbf{A} : L^2(N) \rightarrow L^2(N)$, its pullback via Ψ , denoted by $\Psi^*(\mathbf{A}) : L^2(M) \rightarrow L^2(M)$, is defined by: given any function $f \in L^2(M)$, we obtain another function in $L^2(M)$ which is $\mathbf{A}(f \circ \Psi^{-1}) \circ \Psi$.

LEMMA 3.1. *\mathbf{A} and $\Psi^*(\mathbf{A})$ share the same eigenvalues. The eigenfunctions of $\Psi^*(\mathbf{A})$ are $\{g_i \circ \Psi\}$ where g_i are the eigenfunctions of \mathbf{A} .*

Proof: Take an eigenfunction g_i of \mathbf{A} with eigenvalue ρ , that is, $\mathbf{A}g_i = \rho g_i$. Now consider $f = g_i \circ \Psi$ and consider $\Psi^*(\mathbf{A})f$. We have that

$$\Psi^*(\mathbf{A})f = \mathbf{A}(f \circ \Psi^{-1}) \circ \Psi = \mathbf{A}(g_i) \circ \Psi = \rho g_i \circ \Psi = \rho f.$$

The opposite direction is similar. ■

Since \mathcal{L}_t^N and its pullback share the same spectrum, it suffices to compare \mathcal{L}_t^M with $\Psi^*(\mathcal{L}_t^N)$. The following result will be needed later (the proof is rather standard and is in Appendix A):

CLAIM 3.2. *Given an m -manifold M embedded in \mathbb{R}^d , for small enough $t > 0$, $\int_M e^{-\frac{\|x-y\|^2}{4t}} dy = O(t^{\frac{m}{2}})$.*

LEMMA 3.3. *The L_2 -norm of the difference of \mathcal{L}_t^M and $\Psi^*(\mathcal{L}_t^N)$ is bounded by $\|\mathcal{L}_t^M - \Psi^*(\mathcal{L}_t^N)\| = O(\frac{\delta}{t^{\frac{m}{4}+2}})$.*

Proof: Set $c = \frac{1}{t(4\pi t)^{m/2}}$ and $G_t(x, y) = e^{-\frac{\|x-y\|^2}{4t}}$. Given two points $x, y \in M$, note that $\|\Psi(x) - x\| = O(\delta)$ and $\|\Psi(y) - y\| = O(\delta)$ since M and N are δ -close. Thus

$$|G_t(\Psi(x), \Psi(y)) - G_t(x, y)| = O(\frac{\delta}{t})G_t(x, y).$$

Now, given a function $f : M \rightarrow \mathbb{R}$ and a point $x \in M$, note that $\Psi^*(\mathcal{L}_t^N)f(x) = \mathcal{L}_t^N(f \circ \Psi^{-1}) \circ \Psi(x)$. Setting $g = f \circ \Psi^{-1}$ and $p = \Psi(x)$, we have that:

$$\Psi^*(\mathcal{L}_t^N)f(x) = \mathcal{L}_t^N g(p) = c \int_N G_t(p, q)[g(p) - g(q)]dq.$$

By change of variables, we then obtain:

$$\begin{aligned} & \Psi^*(\mathcal{L}_t^N)f(x) \\ &= c \int_{\Psi^{-1}(N)} G_t(p, \Psi(y))[g(p) - g \circ \Psi(y)]J\Psi|_y dy \\ &= c \int_M G_t(\Psi(x), \Psi(y))[f(x) - f(y)]J\Psi|_y dy, \end{aligned}$$

where $J\Psi|_y$ is the Jacobian of the map Ψ at $y \in M$, and is bounded by $|J\Psi|_y - 1| = O(\delta)$ due to the δ -closeness condition. Comparing this with $\mathcal{L}_t^M f(x)$ (recall Eqn 2.1), we have that:

$$\begin{aligned}
& \left| \Psi^*(\mathcal{L}_t^N)f(x) - \mathcal{L}_t^M f(x) \right| \\
& \leq c \int_M |f(y)(G_t(\Psi(x), \Psi(y))[1 + O(\delta)] - G_t(x, y))| dy \\
& + c \int_M |f(x)(G_t(\Psi(x), \Psi(y))[1 + O(\delta)] - G_t(x, y))| dy \\
& = c \left| \int_M G_t(\Psi(x), \Psi(y))f(y)dy - \int_M G_t(x, y)f(y)dy \right. \\
& \quad \left. + O(\delta) \int_M G_t(\Psi(x), \Psi(y))f(y)dy \right| \\
& + c \int_M |f(x)(G_t(\Psi(x), \Psi(y))(1 + O(\delta)) - G_t(x, y))| dy \\
& \leq c \cdot O\left(\frac{\delta}{t}\right) \left[\int_M G_t(x, y)|f(y)|dy + |f(x)| \int_M G_t(x, y)dy \right] \\
& \leq c \cdot O\left(\frac{\delta}{t}\right) \left[\|f\| \sqrt{\int_M G_t^2(x, y)dy} + |f(x)| \int_M G_t(x, y)dy \right] \\
& \leq O\left(\frac{\delta}{t^{\frac{m}{4}+2}}\right) \|f\| + O\left(\frac{\delta}{t^2}\right) |f(x)|.
\end{aligned}$$

The last but one inequality follows from the fact that $\langle f, g \rangle \leq \|f\| \cdot \|g\|$ for any two functions. The last inequality follows from Claim 3.2. Hence the square of the L_2 -norm of $\Psi^*(\mathcal{L}_t^N)f - \mathcal{L}_t^M f$ is bounded by:

$$\begin{aligned}
\|\Psi^*(\mathcal{L}_t^N)f - \mathcal{L}_t^M f\|^2 &= \int_M [\Psi^*(\mathcal{L}_t^N)f(x) - \mathcal{L}_t^M f(x)]^2 dx \\
&\leq O\left(\frac{\delta^2}{t^{\frac{m}{2}+4}}\right) \int_M (\|f\|^2 + f^2(x) + 2\|f\| \cdot |f(x)|) dx \\
&\leq O\left(\frac{\delta^2}{t^{\frac{m}{2}+4}}\right) (\|f\|^2 \cdot \|\mathbf{1}\| + \|f\|^2 + 2\|f\|^2) \\
&\leq O\left(\frac{\delta^2}{t^{\frac{m}{2}+4}}\right) \|f\|^2,
\end{aligned}$$

where $\mathbf{1}$ is the constant function and $\|\mathbf{1}\| = \text{volume}(M)$. Hence $\|\Psi^*(\mathcal{L}_t^N)f - \mathcal{L}_t^M f\| = O(\delta/t^{\frac{m}{4}+2})\|f\|$ for any function f , where the big-O notation hides terms depending only on the underlying manifold M . The lemma then follows. \blacksquare

This result and Equation (2) from [17] imply that for any eigenvalue $\hat{\omega} \in \text{Spec}(\mathcal{L}_t^N)$, we have that $\text{dist}(\hat{\omega}, \text{Spec}(\mathcal{L}_t^M)) = O(\frac{\delta}{t^{\frac{m}{4}+2}})$. Now switching the role of M and N in Lemma 3.3, we obtain a symmetric result that for any eigenvalue $\hat{\lambda} \in \text{Spec}(\mathcal{L}_t^M)$, we have that $\text{dist}(\hat{\lambda}, \text{Spec}(\mathcal{L}_t^N)) = O(\frac{\delta}{t^{\frac{m}{4}+2}})$. Theorem 2.5 then follows from these two results. We remark that the distance between spectra of \mathcal{L}_t^M and \mathcal{L}_t^N depends not only

on δ , the closeness between M and N , but also on t inversely. Intuitively, this is expected as the parameter t in the functional Laplacian \mathcal{L}_t specifies the width of the Gaussian kernel and thus the range of the region around $x \in M$ influencing $\mathcal{L}_t^M f(x)$. Hence, the larger t is, the stronger the smoothing effect it has, while the smaller t is, the more sensitive the functional Laplacian is to the perturbation of the underlying manifold, which leads to larger error between the corresponding spectra.

Sketch of proof of Theorem 2.6. It is well known that the Laplace operator has only real and isolated eigenvalues with finite multiplicity. We wish to build a one-to-one relationship between $\text{Spec}(\Delta_M)$ and $\text{Spec}(\Delta_N)$ and bound their distances. To achieve this using Theorems 2.2 and 2.5 (recall Diagram 1), there are two main technical issues to be addressed. First, the operator \mathcal{L}_t^X , although bounded and self-adjoint, is not compact. Hence, it may have a continuous spectrum (e.g. all values within an interval are eigenvalues). Second, Theorem 2.5 only bounds the Hausdorff distance between spectra of \mathcal{L}_t^M and \mathcal{L}_t^N , while we wish to obtain a one-to-one relationship between (the lowest) eigenvalues. Below we provide a sketch of how these two issues are addressed; the simple but somewhat technical details can be found in Appendix B.

For the first issue, given an operator T , let $\text{SpecDis}(T)$ denote the set of isolated eigenvalues of T with finite multiplicity. The set

$$\text{SpecEss}(T) = \text{Spec}(T) \setminus \text{SpecDis}(T)$$

is called the *essential spectrum* of T .

CLAIM 3.4. ([2]) *The essential spectrum of \mathcal{L}_t^X is contained in $(\frac{1}{2}t^{-1}, \infty)$. The smallest eigenvalue of \mathcal{L}_t^X is 0, and the discrete spectrum of \mathcal{L}_t^X is contained in the interval $[0, \Theta(\frac{1}{t})]$.*

In other words, even though \mathcal{L}_t^M contains a continuous spectrum, those with low values (smaller than $\frac{1}{2}t^{-1}$) are isolated with finite multiplicity, and can be potentially related to those of \mathcal{L}_t^N in a one-to-one manner. These first few eigenvalues are also what are typically used in practice. As t goes to zero, the interval $[0, \frac{1}{2}t^{-1})$ will contain more and more isolated eigenvalues.

For the second issue, consider the first k eigenvalues $\{\hat{\lambda}_i\}$ of \mathcal{L}_t^M and $\{\hat{\omega}_i\}$ of \mathcal{L}_t^N , in non-decreasing order, where k is an integer such that $\hat{\lambda}_k < \frac{1}{2}t^{-1}$ and $\hat{\omega}_k < \frac{1}{2}t^{-1}$ (i.e. the first k isolated eigenvalues). By using Proposition 6 from [18], we can show that when $\frac{\delta}{t^{\frac{m}{4}+2}}$, the Hausdorff distance between $\text{Spec}(\mathcal{L}_t^M)$ and $\text{Spec}(\mathcal{L}_t^N)$, is small enough, then $|\hat{\lambda}_i - \hat{\omega}_i| = O(\frac{\delta}{t^{\frac{m}{4}+2}})$ for $i \in [1, k]$.

Finally, combining this with Theorem 2.2, we choose $t = \delta^{\frac{4(m+6)}{m^2+14m+56}}$ so that the two convergence rates, between Δ_M (resp. Δ_N) and \mathcal{L}_t^M (resp. \mathcal{L}_t^N), and between \mathcal{L}_t^M and \mathcal{L}_t^N , respectively, are balanced (i.e, $t^{\frac{2}{m+6}} = \frac{\delta}{t^{m/4+2}}$). Theorem 2.6 then follows.

4 Spectra Convergence between Discrete and Continuous Laplacians

In this section, given a mesh K that ε -approximates a smooth compact m -manifold M embedded in \mathbb{R}^d , we relate the spectrum of Δ_M to that of its discrete approximation. By Theorem 2.2, we only need to show spectral convergence between the functional Laplace \mathcal{L}_t^M and the mesh-Laplacian \mathbf{L}_t^K (i.e, Theorem 2.3). Similar to previous section, we will achieve this by showing that the latter converges to the former in some operator norm. The main difference and challenge is that we now need to define the functional space we use to compare the relevant operators more carefully.

Specifically, the discrete Laplacian \mathbf{L}_t^K is a linear operator in \mathbb{R}^n where n is the number of vertices in K ; while \mathcal{L}_t^M is an operator in an infinite dimensional functional space. Hence, in Step 1, we first construct a continuous operator \mathcal{C}_t^K , which (almost) shares the same spectrum as the discrete operator \mathbf{L}_t^K , and which, at the same time, is well-defined in certain a common functional space along with \mathcal{L}_t^M . Next, in Step 2, we bound the operator norm of the difference between \mathcal{C}_t^K and \mathcal{L}_t^M in this space, which will in turn relate their spectra. Below we give a sketch for the procedure, focusing on illustrating the intuitions. Details can be found in Appendix C.

The Sobolev space H_s . The common functional space we use to compare \mathcal{C}_t^K and \mathcal{L}_t^M is the s -th Sobolev space H_s and we will choose $s = \frac{m}{2} + 1$. The norm in H_s is the Sobolev norm $\|g\|_{H_s} = [\sum_{i=0}^s \|g^{(i)}\|^2]^{1/2}$, where $g^{(i)}$ is the i -th weak derivative² of g and $\|\cdot\|$ denotes the standard L_2 norm. The key property of H_s for $s \geq \frac{m}{2} + 1$ that we will need is the following [15] and its corollary.

LEMMA 4.1. ([15]) *Let $f \in H_s(M)$ with $s \geq m/2 + 1$ where m is the intrinsic dimension of the manifold M . Then f is Lipschitz with the Lipschitz constant bounded by $C\|f\|_{H_s}$ for some universal constant C .*

COROLLARY 4.2. *Given any $f \in H_s(M)$ with $s \geq m/2 + 1$, $\|f\|_\infty \leq C'\|f\|_{H_s}$ with some universal constant*

²The weak derivative is a generalization of the derivative of a function f , when f is not necessarily differentiable in the usual sense, and these two notions coincide when f is differentiable. For our purpose, the reader can think of it as the ordinary derivative.

C' depending only on the underlying manifold M .

Proof: The Lipschitz constant of f is bounded by $C\|f\|_{H_s}$ by Lemma 4.1. For any two points $x, y \in M$,

$$\begin{aligned} \left| |f(x)| - |f(y)| \right| &\leq |f(x) - f(y)| \\ &\leq C\|f\|_{H_s} \cdot |x - y| \leq C \cdot \text{Diameter}(M) \cdot \|f\|_{H_s}. \end{aligned}$$

Let $p \in M$ be a point so that $|f(p)| = \min_x |f(x)|$; note that $|f(p)| \leq \|f\| \leq \|f\|_{H_s}$. It then follows that $|f(x)| - |f(p)| \leq C \cdot \text{Diameter}(M) \cdot \|f\|_{H_s}$, implying

$$|f(x)| \leq |f(p)| + C \cdot \text{Diameter}(M) \cdot \|f\|_{H_s} \leq C'\|f\|_{H_s}.$$

The corollary then follows. \blacksquare

From now on, we fix $s = m/2 + 1$. There are two main reasons behind relating the operators of interest in the space $H_s(M)$, instead of using some other spaces, say the space of square integrable functions $L^2(M)$.

(i) We can extend the discrete operator \mathbf{L}_t^K into a well-defined and well-behaved operator in $H_s(M)$. Intuitively, this is not possible in $L^2(M)$, as functions in $L^2(M)$ are not defined pointwise (two functions can be arbitrarily different at a finite set of points while the L_2 -norm of their difference is zero); while at the same time, \mathbf{L}_t^K requires point evaluations (as it is only defined at discrete sample points). Corollary 4.2 guarantees that the point evaluations in $H_{m/2+1}(M)$ are not only defined, but also bounded ($H_i(M)$ is, in fact, a reproducing kernel Hilbert space for $i \geq m/2 + 1$).

(ii) It turns out that we cannot bound the L_2 -norm distance of relevant operators (which is the operator norm in $L^2(M)$). As we will see later, this happens because the Lipschitz constant of the input function appears while bounding the L_2 -norm of the operator difference. Lemma 4.1 says that the Lipschitz constant can be bounded by the s -th Sobolev norm of f , which again suggests that we should use the space $H_s(M)$.

Step 1: Continuous extension for \mathbf{L}_t^K . We extend \mathbf{L}_t^K to an operator $\mathcal{C}_t^K : H_s(M) \rightarrow H_s(M)$ defined as:

$$\mathcal{C}_t^K f(x) = \frac{1}{t(4\pi t)^{m/2}} \sum_{i=1}^n A_i G_t(x, v_i) (f(x) - f(v_i)).$$

Intuitively, we extend the kernel function from an n by n matrix (i.e, $G_t(v_j, v_i)$'s) to a continuous (Gaussian) kernel function defined on $M \times \mathbb{R}^n$. A similar extension was used in [18] to relate the graph Laplacian with the functional Laplacian.

The operator \mathcal{C}_t^K is bounded in $H_s(M)$ and its spectrum has a simple structure (similar to Claim 3.4). Roughly speaking, its *discrete* spectrum is the same as the spectrum of \mathbf{L}_t^K (see Appendix C.1 for more precise

statements). Hence, to bound the difference between $\text{Spec}(\mathbf{L}_t^K)$ and $\text{SpecDis}(\mathcal{L}_t^M)$, it suffices to bound that between $\text{SpecDis}(\mathcal{C}_t^K)$ and $\text{SpecDis}(\mathcal{L}_t^M)$.

Step 2: Relation between \mathcal{C}_t^K and \mathcal{L}_t^M . Let $\mathcal{D} = \mathcal{L}_t^M - \mathcal{C}_t^K$ denote the difference between operators \mathcal{L}_t^M and \mathcal{C}_t^K . We aim to show that $\|\mathcal{D}f\|_{H_s} \leq O(\varepsilon)\|f\|_{H_s}$ for any function $f \in H_s(M)$, which will then imply that $\|\mathcal{D}\|_{H_s} = O(\varepsilon)$. First, the following result bounds the derivatives of the Gaussian kernel function (see Appendix C.2 for the proof).

LEMMA 4.3. $G_t^{(i)}(x, y) = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} O(i^i) \frac{\|x-y\|^{i-2j}}{(2t)^{i-j}} G_t(x, y)$
The derivative is taken with respect to the variable x .

THEOREM 4.1. Set $\mathcal{D}^{(j)}f = (\mathcal{D}f)^{(j)}$ to be the j -th (weak) derivative of the function $\mathcal{D}f$. We have that $\|\mathcal{D}^{(j)}f\| = O(\frac{\varepsilon}{t^{j+2}}(\|f\|_{H_j} + \|f\|_{H_{m/2+1}}))$ for $j \geq 0$, where the big- O notation hides constants exponential in j and dependent on the underlying manifold M .

This implies that $\|\mathcal{L}_t^M - \mathcal{C}_t^K\|_{H_s} = O(\frac{\varepsilon}{t^{s+2}})$ for any $s \geq m/2 + 1$.

Sketch of Proof:

See Appendix C.3 for a detailed proof. Set $c(t)$ to be the constant $\frac{1}{t(4\pi t)^{m/2}}$. One way to interpret the mesh-Laplacian \mathbf{L}_t^K (as well as \mathcal{C}_t^K) is that, for any m -simplex $\sigma \in K$, subdivide it to $m+1$ pieces with equal volume, with each piece σ' represented by one unique vertex, say v , of σ . We refer to the vertex v as the *pivot* p_z of every point z in this portion $\sigma' \subset \sigma$. See the right figure for one such subdivision of a 2-dimensional example, where all points in the shaded region have pivot v . The sampling condition of K implies that $\|z - p_z\| = O(\varepsilon)$. This way, we can rewrite the sum in $\mathcal{C}_t^K f(x)$ as an integral over the underlying space $|K|$ of K ; that is, $\mathcal{C}_t^K f(x) = c(t) \int_{|K|} G_t(x, p_z)(f(x) - f(p_z))dz$. Thus

$$\begin{aligned} \mathcal{D}f(x) &= c(t) \int_M G_t(x, y)(f(x) - f(y))dy \\ &\quad - c(t) \int_{|K|} G_t(x, p_z)(f(x) - f(p_z))dz. \end{aligned}$$

Let $\phi : |K| \rightarrow M$ be the homeomorphism between $|K|$ and M so that K ε -approximates M . By change of variable $z = \phi^{-1}(y)$, we get the following where J_y is

the Jacobian of the map $\phi^{-1} : M \rightarrow |K|$ at $y \in M$.

$$\begin{aligned} \mathcal{D}f(x) &= c(t) \int_M G_t(x, y)(f(x) - f(y))dy \\ &\quad - c(t) \int_M G_t(x, p_y)(f(x) - f(p_y))J_y dy \\ &= c(t) \left[\int_M G_t(x, p_y)f(p_y)J_y dy - \int_M G_t(x, y)f(y)dy \right] \\ &\quad - c(t) \left[\int_M G_t(x, p_y)f(x)J_y dy - \int_M G_t(x, y)f(x)dy \right]. \end{aligned}$$

It then follows that

$$\begin{aligned} \mathcal{D}^{(j)}f(x) &= \\ c(t) &\left[\int_M G_t^{(j)}(x, p_y)f(p_y)J_y dy - \int_M G_t^{(j)}(x, y)f(y)dy \right] \\ &+ c(t) \left[\int_M \sum_{i=0}^j [G_t^{(i)}(x, y)f^{(j-i)}(x)] dy \right. \\ &\quad \left. - \int_M \sum_{i=0}^j [G_t^{(i)}(x, p_y)f^{(j-i)}(x)] J_y dy \right] \end{aligned}$$

Combining this with the fact that $\|y - p_y\| = O(\varepsilon)$, $|J_y - 1| = O(\varepsilon)$ and Lemma 4.3, we can show that :

$$\begin{aligned} |\mathcal{D}^{(j)}f(x)| &\leq c(t) \cdot O\left(\frac{\varepsilon \cdot \text{Lip}_f}{t^{j+1}}\right) \int_M G_t(x, y)dy \\ &\quad + c(t) \cdot O\left(\frac{\varepsilon}{t^{j+1}}\right) \int_M G_t(x, y)f(y)dy \\ &\quad + c(t) \sum_{i=0}^j O\left(\frac{\varepsilon \cdot f^{(j-i)}(x)}{t^{i+1}}\right) \int_M G_t(x, y)dy, \end{aligned}$$

where Lip_f is the Lipschitz constant of the function f , and is bounded by $D\|f\|_{H_s}$ by Lemma 4.1. Furthermore, by Corollary 4.2, $f(y) \leq \|f\|_\infty \leq C'\|f\|_{H_s}$. Combining these with Claim 3.2 we have that:

$$|\mathcal{D}^{(j)}f(x)| \leq O\left(\frac{\varepsilon}{t^{j+2}}\right)\|f\|_{H_s} + \sum_{i=0}^j f^{(j-i)}(x)O\left(\frac{\varepsilon}{t^{i+2}}\right).$$

This implies

$$\begin{aligned} \|\mathcal{D}^{(j)}f\| &\leq O\left(\frac{\varepsilon}{t^{j+2}}\right)\|f\|_{H_s} + O\left(\frac{\varepsilon}{t^{j+2}}\right) \sum_{i=0}^j \|f^{(j-i)}\| \\ &= O\left(\frac{\varepsilon}{t^{j+2}}\right)\|f\|_{H_s} + O\left(\frac{j\varepsilon}{t^{j+2}}\right)\|f\|_{H_j} \\ &= O\left(\frac{\varepsilon}{t^{j+2}}\right)(\|f\|_{H_s} + \|f\|_{H_j}). \end{aligned}$$

The bound on the s -th Sobolev norm for $\mathcal{L}_t^M - \mathcal{C}_t^K$ then follows easily. \blacksquare

Putting everything together. Now, for a fixed t , consider a sequence of meshes $\{K(\varepsilon)\}_{\varepsilon \rightarrow 0}$ that ε -

approximates M and converges to M as ε goes to zero. This induces a sequence of discrete Laplace operators $\{\mathbf{L}_t^{K(\varepsilon)}\}_{\varepsilon \rightarrow 0}$ as well as a sequence of their continuous extensions $\{\mathcal{C}_t^{K(\varepsilon)}\}_{\varepsilon \rightarrow 0}$ in the Sobolev space $H_{m/2+1}(M)$. All these operators have only real eigenvalues. Theorem 4.1 implies that the sequence of operators $\{\mathcal{C}_t^{K(\varepsilon)}\}_{\varepsilon \rightarrow 0}$ converges in operator norm to the functional Laplacian \mathcal{L}_t^M as ε goes to zero. Using Proposition 6 in [18], by a similar argument as the one used in Section 3 (in Appendix B), when ε is small enough, there is a one to one correspondence between the lowest few eigenvalues of $\mathcal{C}_t^{K(\varepsilon)}$ and $\text{SpecDis}(\mathcal{L}_t^M)$ such that the i -th one from $\text{SpecDis}(\mathcal{C}_t^{K(\varepsilon)})$ converges to the i th one from $\text{SpecDis}(\mathcal{L}_t^M)$ as ε goes to zero. Since $\mathcal{C}_t^{K(\varepsilon)}$ shares discrete eigenvalues with $\mathbf{L}_t^{K(\varepsilon)}$ (precise statement in Lemma C.2), this then implies Theorem 2.3. Finally, Theorem 2.4 follows from this and Theorem 2.2. By 4.1, the condition $\frac{\varepsilon}{t^{\frac{m}{2}+3}} \rightarrow 0$ in the limit guarantees that the sequence of continuous extensions $\{\mathcal{C}_t^{K(\varepsilon)}\}_{\varepsilon \rightarrow 0}$ converges to \mathcal{L}_t^M in operator norm.

Proof of Theorem 2.7. Imagine that we have a sequence of manifolds $\{N_\delta\}_{\delta \rightarrow 0}$ that is δ -close to M and δ converges to zero. Now choose $t(\delta) = \Omega(\delta^{\frac{4}{m+8}-\nu})$ for some small constant $\nu > 0$ and denote $\mathcal{L}_{t(\delta)}^{N_\delta}$ by $\mathcal{L}^N(\delta)$. By Lemma 3.3, the sequence of manifolds $(N_\delta)_{\delta \rightarrow 0}$ induces a sequence of operators $(\mathcal{L}^N(\delta))_{\delta \rightarrow 0}$ that converges to \mathcal{L}_t^M in operator norm. Combining Theorem 2.3, Theorem 2.7 then follows from a similar argument as above.

5 Experiments

In this section, we show through experiments that the spectrum of the mesh Laplacian [3] converges to that of the manifold Laplacian, is robust, and changes smoothly with smooth deformation of a surface. For all our experiments, we normalize the input surface to diameter 1. We use the code from Belkin et al. [3] to compute the mesh-Laplacian, and use MATLAB[®] to find its first 300 eigenvalues and eigenvectors.

To demonstrate the convergence behavior, we consider a sequence of increasingly denser meshes approximating a unit sphere, for which we can obtain the ground truth. We use an adaptive t , which becomes smaller as the meshes become denser. The results are shown in Figure 2, where we plot the error of each of the first 300 eigenvalues / eigenfunctions (x -axis is the index of the eigenvalue/eigenfunction). In (a) we plot for each i , the difference $|\lambda_i - \lambda_i^D|$, where λ_i and λ_i^D are the i th eigenvalue of the manifold and mesh Laplacians, respectively. In (b) we plot the error in eigenvectors. Specif-

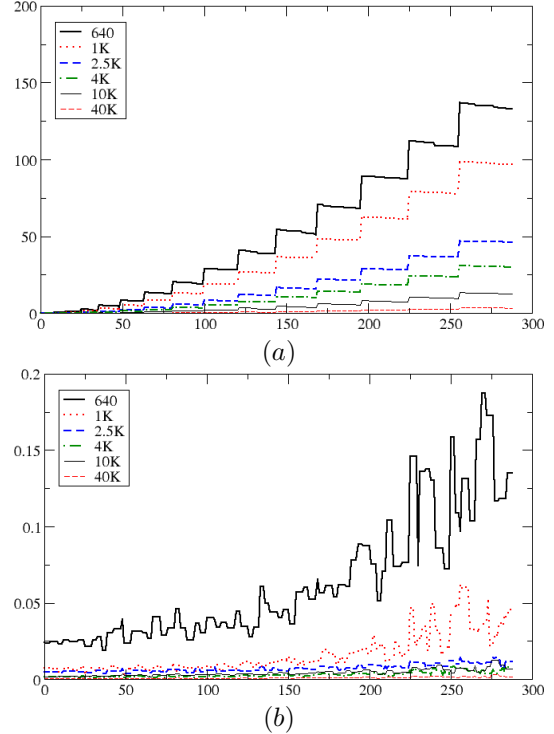


Figure 2: Errors in the (a) eigenvalues and (b) eigenvectors of discrete Laplacian of meshes of unit sphere with increasing number of vertices.

ically, note that the restriction of each ground truth eigenfunction ϕ_i to the vertices of the mesh gives us a vector $\hat{\phi}_i$. We compute the error as the L_2 -norm distance between $\hat{\phi}_i$ and the corresponding discrete eigenvector of the mesh Laplacian. If an eigenvalue has multiplicity more than 1, we project the discrete eigenvector into the eigenspace spanned by the restricted eigenfunctions corresponding to that eigenvalue and return the error as distance between this vector and its projection. As we can see, the eigenvalues and eigenvectors converge to ground truth as the sampling density increases.

Next, we show that, with a fixed t , the mesh-Laplacian is robust against changes in the sampling density, noise, and quality of sampling. Here we use a more interesting genus 3 surface (see Figure 3), and plot the spectra of different meshes in the bottom picture, where x -axis is the index of each eigenvalue, and y -axis is the value. All these curves are close, indicating that the discrete Laplacian spectra are resilient to these changes.

For nearly isometric deformations, we use various poses of a human figure (Figure 4), and show that the discrete Laplacian spectrum is robust against such deformations. Finally, we investigate how the discrete Laplacian spectrum changes as the manifold undergoes

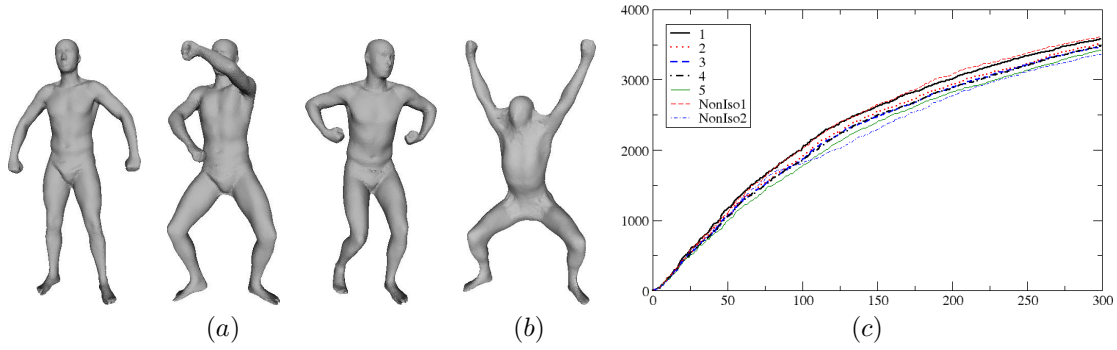


Figure 4: (a) Some near-isometric deformations of a human. (b) An example of non-isometric deformation. (c) Comparison of spectra computed from five isometric and two non-isometric deformations.

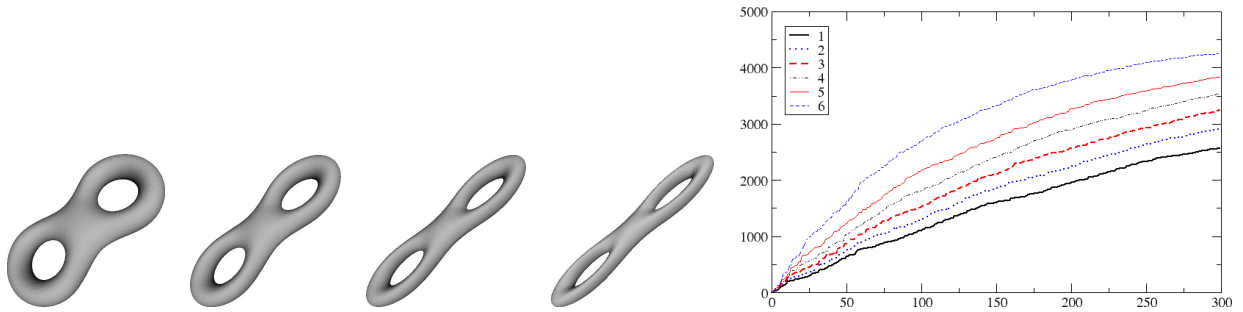


Figure 5: Snapshots of continuous deformation of an eight loop and plot of spectra of corresponding meshes.

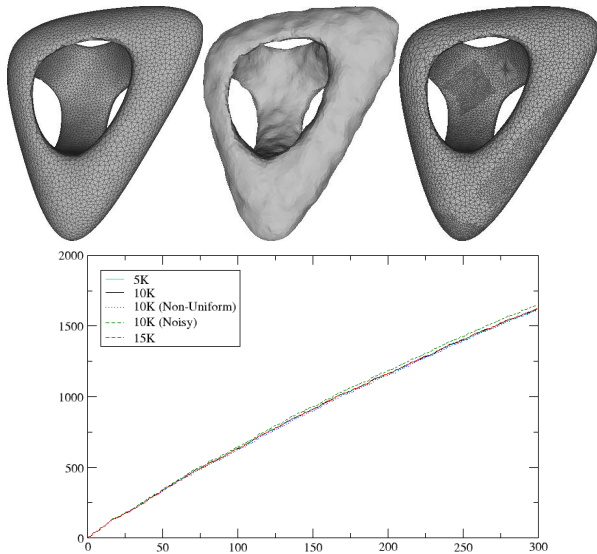


Figure 3: Original, noisy, and non-uniform meshes for the same genus 3 surface. Bottom : comparison of their eigenvalues.

larger deformations. Specifically, we continuously deform a figure-eight loop and plot the corresponding discrete Laplacian spectra. See Figure 5 and note the spectrum also changes continuously with the deformations.

6 Conclusion and Discussion

This paper provides the first result showing that eigenvalues of a certain discrete Laplace operator [3] approximated from a general mesh in d -dimensional space converge to those of the manifold Laplacian as the mesh converges to a smooth manifold. It also shows that the spectrum of this discrete mesh-Laplacian is stable when the smooth manifold is perturbed, which is demonstrated by experimental studies. This helps to provide theoretical guarantees for applications using the mesh-Laplace operator.

In this paper, we only focus on the eigenvalues of the Laplace operator. Another important family of eigen-structures is the set of Laplacian eigenfunctions. Indeed, these eigenfunctions have been widely used in spectral mesh processing applications. We believe that similar convergence results can be obtained for the eigenfunctions as well³ using the separation gap

³To be more careful, for eigenvalues with multiplicity more than one, we should consider the eigenspace spanned by the corresponding eigenfunctions.

between consecutive distinct eigenvalues. Experimental results also show that eigen-spaces are stable. We leave the precise statement and formal proof of stability for eigenfunctions as an immediate future work.

Another future work is to investigate similar problems for discrete point-cloud Laplace operator, constructed from a set of unorganized points sampled from a hidden manifold. Such input is common as demonstrated by the plethora of high dimensional data in various scientific and engineering applications. As a result, many recent work focus on processing point data for spectral shape analysis. It appears that results from this paper can be extended to the point-clouds Laplacian proposed in [8] when the input points is a so-called (ε, η) -sample of a manifold M ; namely, (i) for every point $p \in M$ there is a sample point at most $\varepsilon\rho(M)$ away, where $\rho(M)$ is the reach of M , and (ii) no two sample points are within distance $\eta\rho(M)$. It will be interesting to see whether similar results can be established for the more general ε -sampling without the η -sparsity condition.

Finally, most of our results only show convergence instead of explicitly bounding the error between the discrete and true Laplacian spectra. An explicit error bound not only helps the theoretical understanding of discrete mesh Laplacian but also has practical implications. It will be interesting to explore this direction.

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A Proof for Claim 3.2

Choose $r = (t)^{1/4} \leq \rho/2$ as a constant small enough, where ρ is the reach of the manifold M . Let B be the ball centered at point x with radius r , and M_B the intersection between B and M . First, observe that $e^{-\frac{r^2}{4t}} \leq o(t^\alpha)$ for any $\alpha > 0$ when t is small enough, as

$$\lim_{t \rightarrow 0} e^{-\frac{r^2}{4t}} / t^\alpha = \lim_{t \rightarrow 0} e^{-\frac{1}{4\sqrt{t}}} / t^\alpha = 0.$$

It then follows that

$$(A.1) \quad \int_{M \setminus M_B} e^{-\frac{\|x-y\|^2}{4t}} dy \leq \text{Vol}(M) e^{-\frac{r^2}{4t}} = o(t^{m/2}).$$

On the other hand, consider the map from M_B to T_x , where T_x is the tangent space at x of M . Obviously, T_x is a m -dimensional subspace. Consider the projection map $\phi : M_B \rightarrow T_x$. For $r < \rho/2$, ϕ is injective. It is shown in [3] that the Jacobian of ϕ at any $y \in M_B$ is bounded by $1 + O(r^2/\rho^2)$. Same bound holds for the Jacobian of ϕ^{-1} for any $z \in \phi(M_B)$. This also implies that

$$\|\phi(y) - x\| \geq (1 - O(r^2/\rho^2))\|y - x\|.$$

Applying change of variables, we have:

$$\begin{aligned} \int_{M_B} e^{-\frac{\|x-y\|^2}{4t}} dy &= \int_{\phi(M_B)} e^{-\frac{\|x-\phi^{-1}(z)\|^2}{4t}} J_{\phi^{-1}}(z) dz \\ &\leq \int_{\phi(M_B)} e^{-\frac{(1-O(r^2/\rho^2))\|x-z\|^2}{4t}} (1 + O(\frac{r^2}{\rho^2})) dz \\ &\leq \int_{\phi(M_B)} e^{O(\frac{r^2\|x-z\|^2}{4t})} e^{-\frac{\|x-z\|^2}{4t}} (1 + O(\sqrt{t})) dz \\ &\leq \int_{\phi(M_B)} 2e^{O(\frac{r^4}{4t})} e^{-\frac{\|x-z\|^2}{4t}} dz \\ &\leq O(1) \cdot \int_{\phi(M_B)} e^{-\frac{\|x-z\|^2}{4t}} dz \\ &\leq O(1) \cdot \int_{\mathbb{R}^m} e^{-\frac{\|x-z\|^2}{4t}} dz \leq O(t^{m/2}). \end{aligned}$$

The last inequality follows from Claim 3.1 from [9]. The claim then follows from this and Eqn (A.1).

B Details from Proof for Theorem 2.6

It is well known that the Laplace operator only has real and isolated eigenvalues with finite multiplicity. We wish to build a one-to-one relationship between $\text{Spec}(\Delta_M)$ and $\text{Spec}(\Delta_N)$ and bound their distance. To achieve this using Theorems 2.2 and 2.5 (recall Diagram 1), there are two main technical issues to be addressed. First, the operator \mathcal{L}_t^X , although bounded and self-adjoint, is not compact. Hence, it may have non-isolated a continuous spectrum (e.g, all values within an interval

are eigenvalues). Second, Theorem 2.5 only bounds the Hausdorff distance between spectra of \mathcal{L}_t^M and \mathcal{L}_t^N , while we wish to obtain a one-to-one relationship between (their lowest) eigenvalues.

For the first issue, given an operator T , recall that $\text{SpecDis}(T)$ denotes the set of isolated eigenvalues of T with finite multiplicity, and $\text{SpecEss}(T) = \text{Spec}(T) \setminus \text{SpecDis}(T)$ is the so-called essential spectrum of T . Claim 3.4 was shown in [2, 18]. We provide an intuition here: Set $c(t) = \frac{1}{t(4\pi t)^{m/2}}$ and

$G_t(x, y) = e^{-\frac{\|x-y\|^2}{4t}}$. It turns out that the operator \mathcal{L}_t^X can be rewritten as $\mathcal{L}_t^X = \mathbf{M}_t^X - \mathbf{I}_t^X$, where $\mathbf{M}_t^X f(x) = g(x) \cdot f(x)$ with $g(x) = c(t) \int_X G_t(x, y) dy$, and $\mathbf{I}_t^X f(x) = c(t) \int_X G_t(x, y) f(y) dy$. In other words, \mathbf{M}_t^X is a multiplication operator and \mathbf{I}_t^X is an integral operator. It is easy to verify that both are self-adjoint in $L^2(X)$, and the former is bounded while the latter is compact. It then follows that the essential spectrum of \mathcal{L}_t^X coincide with the range of the function $g(\cdot)$ (i.e, $[\inf g(x), \sup g(x)]$). The range of this function $g(\cdot)$ was shown in [2] and Claim 3.4 thus follows.

Claim 3.4 states that, even though \mathcal{L}_t^M contains continuous spectrum, those with low values (smaller than $\frac{1}{2}t^{-1}$) are isolated with finite multiplicity, and can be potentially related to those of \mathcal{L}_t^N in a one-to-one manner. These top few eigenvalues are also what are typically used in practice. As t goes to zero, the interval $[0, \frac{1}{2}t^{-1})$ will contain more and more isolated eigenvalues.

For the second issue, consider the first k eigenvalues $\{\hat{\lambda}_i\}$ of \mathcal{L}_t^M and $\{\hat{\omega}_i\}$ of \mathcal{L}_t^N , in non-decreasing order, where k is an integer such that $\hat{\lambda}_k < \frac{1}{2}t^{-1}$ and $\hat{\omega}_k < \frac{1}{2}t^{-1}$ (i.e, the first k isolated eigenvalues). Theorem 2.2 from [2] states that for each $i < k$, $|\lambda_i - \hat{\lambda}_i| = O(t^{\frac{2}{m+6}})$. In other words, the first few eigenvalues from \mathcal{L}_t^M one-to-one correspond to the first few eigenvalues from Δ_M . The same statement holds for the lowest eigenvalues $\{\omega_i\}$ for Δ_N and $\{\hat{\omega}_i\}$ for \mathcal{L}_t^N .

Now we wish to also establish a one-to-one correspondence between (lowest) eigenvalues $\{\hat{\lambda}_i\}$ and $\{\hat{\omega}_i\}$. Imagine a sequence of manifolds $\{N(\delta)\}_{\delta \rightarrow 0}$ that converges to M , where $N(\delta)$ is δ -close to M . This induces a sequence of functional Laplacians $\{\mathcal{L}_t^{N(\delta)}\}_{\delta \rightarrow 0}$, and Lemma 3.3 states that this sequence of functional Laplacians converges in operator norm to \mathcal{L}_t^M as δ goes to zero. It then follows from Proposition 6 in [18] that, for any isolated eigenvalue $\hat{\lambda} \in \text{SpecDis}(\mathcal{L}_t^M)$ with multiplicity m , and any open interval $I \in \mathbb{R}$ containing $\hat{\lambda}$ but no other eigenvalue from $\text{SpecDis}(\mathcal{L}_t^M)$, there exists some $\delta_0 > 0$ such that for any $\delta < \delta_0$, exactly m (not necessarily distinct) eigenvalues of $\mathcal{L}_t^{N(\delta)}$ are contained in I . A similar result, in fact, holds for a finite set of

consecutive isolated eigenvalues from $\text{SpecDis}(\mathcal{L}_t^M)$.

Now, imagine we plot the first k isolated eigenvalues $\hat{\lambda}_i$ of \mathcal{L}_t^M on a real line. See the right figure where each empty dot is a distinct eigenvalue of \mathcal{L}_t^M with multiplicity. For each one, we choose an open interval around it as shown in the figure (so their closures partition the line). Proposition 6 in [18] says that eventually (when δ is small enough), for the i th eigenvalue, only exactly m_i number of eigenvalues from $\text{Spec}(\mathcal{L}_t^{N(\delta)})$ will fall in the interval around it, where m_i is the multiplicity of $\hat{\lambda}_i$. (The right figure shows an example where $m_i = 3$, and the black dots represents eigenvalues of $\mathcal{L}_t^{N(\delta)}$.) This idea, combined with the one-to-one correspondence result between λ_i and $\hat{\lambda}_i$, eventually implies that when t is small enough and when $\frac{\delta}{t^{\frac{m}{4}+2}}$ is smaller than the separation gap between two consecutive λ_i s (which is a quantity depending only on the underlying manifold M when t is small enough), there is a one to one correspondence between $\hat{\lambda}_i$ and $\hat{\omega}_i$ and their distance is $O(\frac{\delta}{t^{\frac{m}{4}+2}})$. Specifically, each empty dot in the real line will be a set of m_i number of $\hat{\lambda}$ s clustered within a ball of radius $O(t^{\frac{2}{m+6}})$ where m_i is the multiplicity of the eigenvalue λ_i of Δ_M , and Proposition 6 in [18] states that there will be exactly m_i number of $\hat{\omega}$ s in the corresponding interval.

Finally, by choosing t so that the two convergence rates, between Δ_M (resp. Δ_N) to \mathcal{L}_t^M (resp. \mathcal{L}_t^N) and between \mathcal{L}_t^M and \mathcal{L}_t^N , are balanced, that is, $t^{\frac{2}{m+6}} = \frac{\delta}{t^{\frac{m}{4}+2}}$, we obtain the result in Theorem 2.6. The various conditions in these theorems on the value of the eigenvalues are to ensure that the eigenvalues fall in the discrete spectrum for the functional Laplacian. Their existence does not matter for those lowest eigenvalues, which are the interesting ones in practice.

C Details from Section 4

Below are several details missing from Section 4.

C.1 Step 1: Continuous Extension for \mathbf{L}_t^K

We define operator $\mathcal{C}_t^K : H_s(M) \rightarrow H_s(M)$ as:

$$\mathcal{C}_t^K f(x) := \frac{1}{t(4\pi t)^{m/2}} \sum_{i=1}^n A_i G_t(x, v_i) (f(x) - f(v_i)).$$

Intuitively, we extend the kernel function from an n by n matrix (i.e., $G_t(v_j, v_i)$'s) to a continuous (Gaussian) kernel function defined on $M \times \mathbb{R}^n$. A similar extension was used in [18] to relate the graph Laplacian with the functional Laplacian.

Roughly speaking, there is a ‘‘one-to-one’’ correspondence between the eigenvalues (as well as eigenfunctions) of the operator \mathcal{C}_t^K and those of the discrete operator \mathbf{L}_t^K . To make this correspondence more precise, set a function $d_K : M \rightarrow \mathbb{R}$ as

$$d_K(x) = \frac{1}{t(4\pi t)^{m/2}} \sum_{i=1}^n A_i G_t(x, v_i)$$

and define the multiplication operator $S_K : H_s(M) \rightarrow H_s(M)$ as $S_K f(x) = d_K(x) f(x)$. Set $W_K : H_s(M) \rightarrow H_s(M)$ as

$$W_K f(x) = \frac{1}{t(4\pi t)^{m/2}} \sum_{i=1}^n [A_i G_t(x, v_i) f(v_i)].$$

It is easy to check that the operator $\mathcal{C}_t^K = S_K - W_K$. In space $H_s(M)$ where point evaluation is bounded (recall $H_s(M)$ is a reproducing kernel Hilbert space), S_K is a bounded multiplication operator and W_K is a compact operator [18]; implying that \mathcal{C}_t^K is bounded.

Unfortunately, the spectrum of \mathcal{C}_t^K may contain continuous spectrum. However, similar to the case of \mathcal{L}_t^X in Section 3, since W_K is compact, it turns out that $\text{SpecEss}(\mathcal{C}_t^K) = \text{SpecEss}(S_K) = \text{range}(d_K)$, where $\text{range}(d_K)$ is the range of the function d_K (i.e., $\text{range}(d_K) = [\inf_x d_K(x), \sup_x d_K(x)]$). Lemma C.1 can then be derived by results from [18] and Lemma C.2 follows from elementary calculations:

LEMMA C.1. *The essential spectrum of \mathcal{C}_t^K coincide with the range of the function d_K . For ε and t small enough, $\text{range}(d_K)$ (and thus $\text{SpecEss}(\mathcal{C}_t^K)$) is contained in $(\frac{1}{2}t^{-1}, \infty)$. The discrete spectrum of \mathcal{C}_t^K contains finite number of real eigenvalues, and is contained in the interval $[0, \Theta(\frac{1}{t})]$.*

LEMMA C.2. 1. *If ρ is an eigenfunction of \mathcal{C}_t^K with arbitrary eigenvalue λ , then the n -vector $\hat{\rho} = [\rho(v_1), \dots, \rho(v_n)]^T \in \mathbb{R}^n$ is an eigenvector of \mathbf{L}_t^K with eigenvalue λ .*

2. *If $\lambda \notin \text{range}(d_K) = \text{SpecEss}(\mathcal{C}_t^K)$ is an eigenvalue with multiplicity m , and ρ_1, \dots, ρ_m are the corresponding eigenfunctions, then \mathbf{L}_t^K has an eigenvalue λ also with multiplicity m , with the set of n -vectors $\hat{\rho}_1, \dots, \hat{\rho}_m$ being the corresponding m eigenvectors.*

3. *If $\lambda \notin \text{range}(d_K)$ is an eigenvalue for \mathbf{L}_t^K with multiplicity m , and $\hat{\rho}_1, \dots, \hat{\rho}_m$ being the corresponding m eigenvectors, then λ is an eigenvalue of \mathcal{C}_t^K with multiplicity m , corresponding to a set of eigenfunctions ρ_1, \dots, ρ_m such that*

$$\rho_i(x) = \frac{1}{t(4\pi t)^{m/2}} \cdot \frac{\sum_{j=1}^n A_j G_t(x, v_j) \hat{\rho}_i[j]}{d_K(x) - \lambda}.$$

These results state that the interesting eigenvalues (i.e, with lowest values) are isolated with finite multiplicity, and that there is a one-to-one correspondence between such eigenvalues of \mathcal{C}_t^K and of \mathbf{L}_K .

C.2 Proof for Lemma 4.3

Recall that $G_t(x, y) = e^{-\frac{\|x-y\|^2}{4t}}$. We will prove the following statement by induction (which immediately implies Lemma 4.3).

$$\begin{aligned} G_t^{(i)}(x, y) &= \sum_{j=0}^{\lfloor i/2 \rfloor} c_{j,i} \frac{\|x-y\|^{i-2j}}{(2t)^{i-j}} G_t(x, y) \text{ where} \\ c_{0,0} &= 1, \quad c_{0,i} = -c_{0,i-1}, \\ c_{j,i} &= (i-2j+1)c_{j-1,i-1} - c_{j,i-1}, \quad 0 < j \leq \lfloor i/2 \rfloor, \\ c_{j,i} &= 0, \text{ otherwise.} \\ \text{and } |c_{j,i}| &= O((i+1)^i). \end{aligned}$$

Now for the base case $i = 1$, we have

$$\begin{aligned} G_t^{(1)}(x, y) &= -\frac{\|x-y\|}{2t} G_t(x, y) = c_{0,1} \frac{\|x-y\|}{2t} G_t(x, y) \\ &= \sum_{j=0}^{\lfloor i/2 \rfloor} c_{j,i} \frac{\|x-y\|^{i-2j}}{(2t)^{i-j}} G_t(x, y). \end{aligned}$$

Thus the claim holds. For $G_t^{(i+1)}(x, y)$, we need to consider two cases - when i is odd and when i is even.

Case 1: i is odd: Inductive hypothesis states

$$\begin{aligned} G_t^{(i)}(x, y) &= G_t(x, y) \left[c_{0,i} \frac{\|x-y\|^i}{(2t)^i} + c_{1,i} \frac{\|x-y\|^{i-2}}{(2t)^{i-1}} \right. \\ &\quad \left. + \dots + c_{\frac{i-1}{2},i} \frac{\|x-y\|}{(2t)^{\frac{i+1}{2}}} \right] \end{aligned}$$

We then have:

$$\begin{aligned} G_t^{(i+1)}(x, y) &= G_t(x, y) \left[ic_{0,i} \frac{\|x-y\|^{i-1}}{(2t)^i} - c_{0,i} \frac{\|x-y\|^{i+1}}{(2t)^{i+1}} \right. \\ &\quad \left. + (i-2)c_{1,i} \frac{\|x-y\|^{i-3}}{(2t)^{i-1}} - c_{1,i} \frac{\|x-y\|^{i-1}}{(2t)^i} \right. \\ &\quad \left. + \dots + c_{\frac{i-1}{2},i} \frac{1}{(2t)^{\frac{i+1}{2}}} - c_{\frac{i-1}{2},i} \frac{\|x-y\|^2}{(2t)^{\frac{i+3}{2}}} \right] \end{aligned}$$

Grouping terms together, we get

$$\begin{aligned} G_t^{(i+1)}(x, y) &= G_t(x, y) \left[-c_{0,i} \frac{\|x-y\|^{i+1}}{(2t)^{i+1}} + [ic_{0,i} - c_{1,i}] \frac{\|x-y\|^{i-1}}{(2t)^i} \right. \\ &\quad \left. + [(i-2)c_{1,i} - c_{2,i}] \frac{\|x-y\|^{i-3}}{(2t)^{i-1}} + \dots \right. \\ &\quad \left. + [3 \cdot c_{\frac{i-3}{2},i} - c_{\frac{i-1}{2},i}] \frac{\|x-y\|^2}{(2t)^{\frac{i+3}{2}}} + c_{\frac{i-1}{2},i} \frac{1}{(2t)^{\frac{i+1}{2}}} \right] \\ &= \sum_{j=0}^{\frac{i+1}{2}} c_{j,i+1} \frac{\|x-y\|^{i+1-2j}}{(2t)^{i+1-j}} G_t(x, y) \end{aligned}$$

where

$$\begin{aligned} c_{0,i+1} &= -c_{0,i}, \\ c_{j,i+1} &= ((i+1) - 2j + 1)c_{j-1,i} - c_{j,i}, \quad 0 < j \leq \frac{i+1}{2}, \\ c_{j,i+1} &= 0, \text{ otherwise.} \end{aligned}$$

Case 2: i is even: Can be shown by a similar argument to Case 1.

Finally, to bound the value of $c_{j,i}$, note that $c_{j,i} = (i-2j+1)c_{j-1,i-1} - c_{j,i-1} \leq i|c_{j-1,i-1}| - c_{j,i-1}$. One can then easily use the substitution method to show that $c_{j,i} = O((i+1)^i)$.

We remark that for simplicity, here we proceed as if x is a one-dimensional variable. In general, $x \in M$ is of m -dimension and one needs to compute the derivative w.r.t all mixed terms of coordinates. This will increase the bound by a factor that is exponential in m , but will not affect our final results.

C.3 Detailed Proof for Theorem 4.1

Recall that $\mathcal{D} = \mathcal{L}_t^M - \mathcal{C}_t^K$. Set $c(t)$ to be the constant $\frac{1}{i(4\pi t)^{m/2}}$, and let $|K|$ denote the underlying space of the mesh K . One way to interpret the mesh-Laplacian \mathbf{L}_t^K (as well as \mathcal{C}_t^K) is that, for any m -dimensional simplex $\sigma \in K$, subdivide it to $m+1$ equal volume portions, with every portion σ' being represented by a different vertex, say v , of σ . We refer to the vertex v as the *pivot* p_z of every point z in this portion $\sigma' \subset \sigma$. The sampling condition of K implies that $\|z - p_z\| = O(\varepsilon)$. This way, we can rewrite

$$\mathcal{C}_t^K f(x) = c(t) \int_{|K|} G_t(x, p_z) (f(x) - f(p_z)) dz,$$

and thus

$$\begin{aligned} \mathcal{D}f(x) &= c(t) \int_M G_t(x, y) (f(x) - f(y)) dy \\ &\quad - c(t) \int_{|K|} G_t(x, p_z) (f(x) - f(p_z)) dz. \end{aligned}$$

Let $\phi : |K| \rightarrow M$ be the homeomorphism between $|K|$ and M so that K ε -approximates M . By change of variable $z = \phi^{-1}(y)$, we get the following where J_y is the Jacobian of the map $\phi^{-1} : M \rightarrow |K|$ at $y \in M$.

$$\begin{aligned} \mathcal{D}f(x) &= c(t) \int_M G_t(x, y)(f(x) - f(y))dy \\ &\quad - c(t) \int_M G_t(x, p_y)(f(x) - f(p_y))J_y dy \\ &= c(t) \left[\int_M G_t(x, p_y)f(p_y)J_y dy - \int_M G_t(x, y)f(y)dy \right] \\ &\quad - c(t) \left[\int_M G_t(x, p_y)f(x)J_y dy - \int_M G_t(x, y)f(x)dy \right]. \end{aligned}$$

It then follows that

$$\begin{aligned} \text{(C.2)} \quad \mathcal{D}^{(j)}f(x) &= \\ c(t) &\left[\int_M G_t^{(j)}(x, p_y)f(p_y)J_y dy - \int_M G_t^{(j)}(x, y)f(y)dy \right] \\ &+ c(t) \left[\int_M \sum_{i=0}^j [G_t^{(i)}(x, y)f^{(j-i)}(x)] dy \right. \\ &\quad \left. - \int_M \sum_{i=0}^j [G_t^{(i)}(x, p_y)f^{(j-i)}(x)] J_y dy \right] \end{aligned}$$

On the other hand, since $\|y - p_y\| = O(\varepsilon)$, we have

$$\begin{aligned} \text{(C.3)} \quad \|x - y\|^j - O(\varepsilon\|x - y\|^{j-1}) &\leq \|x - p_y\|^j \\ &\leq \|x - y\|^j + O(\varepsilon\|x - y\|^{j-1}) \quad \text{and} \end{aligned}$$

$$\begin{aligned} \text{(C.4)} \quad (1 - O(\varepsilon/t))G_t(x, y) &\leq G_t(x, p_y) \\ &\leq (1 + O(\varepsilon/t))G_t(x, y) \end{aligned}$$

Let G_t to denote $G_t(x, y)$ and $\alpha = O(\varepsilon^i)$ to simplify

exposition. Using Lemma 4.3, we have:

$$\begin{aligned} &|G_t^{(i)}(x, y) - G_t^{(i)}(x, p_y)| \\ &\leq \sum_{j=0}^{\lfloor i/2 \rfloor} |c_{j,i}| \left| \frac{\|x - p_y\|^{i-2j}}{t^{i-j}} G_t(x, p_y) - \frac{\|x - y\|^{i-2j}}{t^{i-j}} G_t \right| \\ &\leq \sum_{j=0}^{\lfloor i/2 \rfloor} \alpha \left[(1 + O(\frac{\varepsilon}{t})) \frac{\|x - p_y\|^{i-2j}}{t^{i-j}} G_t - \frac{\|x - y\|^{i-2j}}{t^{i-j}} G_t \right] \\ &\quad \text{(Using Eqn C.4)} \\ &\leq \sum_{j=0}^{\lfloor i/2 \rfloor} \alpha \left[(1 + O(\frac{\varepsilon}{t})) G_t \left(\frac{\|x - y\|^{i-2j}}{t^{i-j}} + \frac{\varepsilon\|x - y\|^{i-2j-1}}{t^{i-j}} \right) \right. \\ &\quad \left. - \frac{\|x - y\|^{i-2j}}{t^{i-j}} G_t \right] \quad \text{(Using Eqn C.3)} \\ &\leq \sum_{j=0}^{\lfloor i/2 \rfloor} \alpha G_t \left[O(\frac{\varepsilon}{t}) \frac{\|x - y\|^{i-2j}}{t^{i-j}} + O(\varepsilon) \frac{\|x - y\|^{i-2j-1}}{t^{i-j}} \right] \\ &\leq \sum_{j=0}^{\lfloor i/2 \rfloor} \alpha O(\frac{\varepsilon D^{i-2j}}{t^{i-j+1}}) G_t \leq O(\frac{i D^i \varepsilon}{t^{i+1}}) G_t(x, y), \end{aligned}$$

where D is the diameter of the manifold M . Furthermore, as $\|x - y\| \leq D$ and $G_t(x, y) \leq 1$,

$$G_t^{(i)}(x, y) = O(\frac{i \cdot D^i}{t^i} G_t(x, y)) = O(\frac{1}{t^i}) G_t(x, y)$$

Combined with Eqn (C.2) and that $|J_y - 1| = O(\varepsilon)$, we have that (again, G_t denotes $G_t(x, y)$):

$$\begin{aligned} &\frac{1}{c(t)} |\mathcal{D}^{(j)}f(x)| \\ &\leq \left| \int_M [G_t^{(j)}(x, p_y)f(p_y)J_y - G_t^{(j)}f(y)] dy \right| \\ &\quad + \sum_{i=0}^j \left| f^{(j-i)}(x) \int_M [G_t^{(i)} - G_t^{(i)}(x, p_y)J_y] dy \right| \\ &\leq \left| \int_M [(1 + O(\varepsilon)) [O(\frac{\varepsilon G_t}{t^{j+1}}) + G_t^{(j)}] f(p_y) - G_t^{(j)} f(y)] dy \right| \\ &\quad + \sum_{i=0}^j \left| f^{(j-i)}(x) \int_M [G_t^{(i)} - (1 + O(\varepsilon)) (O(\frac{\varepsilon G_t}{t^{i+1}}) + G_t^{(i)})] dy \right| \\ &\leq \left| \int_M O(\frac{\varepsilon G_t}{t^{j+1}}) f(p_y) dy + \int_M G_t^{(j)} [(1 + O(\varepsilon)) f(p_y) - f(y)] dy \right| \\ &\quad + \sum_{i=0}^j \left| f^{(j-i)}(x) \int_M [O(\frac{\varepsilon}{t^{i+1}}) G_t - O(\varepsilon) G_t^{(i)}] dy \right| \\ &\leq \left| \int_M O(\frac{\varepsilon}{t^{j+1}}) G_t [O(\varepsilon) \text{Lip}_f + f(y)] dy \right| \\ &\quad + \left| \int_M O(\frac{\varepsilon G_t}{t^j}) [\text{Lip}_f + f(y)] dy \right| + \sum_{i=0}^j f^{(j-i)}(x) \int_M O(\frac{\varepsilon G_t}{t^{i+1}}) dy \end{aligned}$$

Hence

$$\begin{aligned}
& | \mathcal{D}^{(j)} f(x) | \\
& \leq c(t) O\left(\frac{\varepsilon \text{Lip}_f}{t^{j+1}}\right) \int_M G_t dy + c(t) O\left(\frac{\varepsilon}{t^{j+1}}\right) \left| \int_M G_t f(y) dy \right| \\
& \quad + c(t) \sum_{i=0}^j f^{(j-i)}(x) \cdot O\left(\frac{\varepsilon}{t^{i+1}}\right) \int_M G_t dy
\end{aligned}$$

where Lip_f is the Lipschitz constant of the function f , which is bounded by $C\|f\|_{H_s}$ by Lemma 4.1. Furthermore, by Corollary 4.2, $f(y) \leq \|f\|_\infty \leq C'\|f\|_{H_s}$. Combining these with Claim 3.2 we have that:

$$| \mathcal{D}^{(j)} f(x) | \leq O\left(\frac{\varepsilon}{t^{j+2}}\right) \|f\|_{H_s} + \sum_{i=0}^j f^{(j-i)}(x) O\left(\frac{\varepsilon}{t^{i+2}}\right).$$

This implies

$$\begin{aligned}
\| \mathcal{D}^{(j)} f \| & \leq O\left(\frac{\varepsilon}{t^{j+2}}\right) \|f\|_{H_s} + O\left(\frac{\varepsilon}{t^{j+2}}\right) \sum_{i=0}^j \|f^{(j-i)}\| \\
& = O\left(\frac{\varepsilon}{t^{j+2}}\right) \|f\|_{H_s} + O\left(\frac{j\varepsilon}{t^{j+2}}\right) \|f\|_{H_j} \\
& = O\left(\frac{\varepsilon}{t^{j+2}}\right) (\|f\|_{H_s} + \|f\|_{H_j}).
\end{aligned}$$

The bound on the s -th Sobolev norm for $\mathcal{L}_t^M - \mathcal{C}_t^K$ then follows easily.