

Homology Groups

Homology groups are algebraic tools to quantify topological features in a space. It does not capture all topological aspects of a space in the sense that two spaces with the same homology groups may not be topologically equivalent. However, two spaces that are topologically equivalent must have isomorphic homology groups. It turns out that the homology groups are computationally tractable in many cases, thus making them more attractive in topological data analysis.

11 Chains

Let \mathcal{K} be a simplicial complex. A p -chain c in \mathcal{K} is a formal sum of p -simplices added with some coefficients, that is, $c = \sum_{i=1}^k \alpha_i \sigma_i$ where σ_i are the p -simplices and α_i are the coefficients. Two p -chains $c = \sum \alpha_i \sigma_i$ and $c' = \sum \alpha'_i \sigma_i$ can be added to obtain another p -chain

$$c + c' = \sum_{i=1}^k (\alpha_i + \alpha'_i) \sigma_i.$$

These additions can be integer additions where the coefficients are integers. So, for example, two 1-chains (edges) we get

$$(2e_1 + 3e_2 + 5e_3) + (e_1 + 7e_2 + 6e_4) = 3e_1 + 10e_2 + 5e_3 + 6e_4.$$

In our case, we will mostly be interested in \mathbb{Z}_2 -additions. This means that the coefficients can only be 0 or 1 which follow the modulo-2 additions $0 + 0 = 0$, $0 + 1 = 1$, and $1 + 1 = 0$. Under \mathbb{Z}_2 -additions, for example, we have

$$(e_1 + e_3 + e_4) + (e_1 + e_2 + e_3) = e_2 + e_4.$$

Observe that p -chains with \mathbb{Z}_2 -coefficients can be treated as sets: the chain $e_1 + e_3 + e_4$ is the set $\{e_1, e_3, e_4\}$, and \mathbb{Z}_2 -addition between two chains is the symmetric difference between the corresponding sets.

From now on, unless specified otherwise, we will consider all chain additions to be \mathbb{Z}_2 -additions. One should keep in mind that one can have parallel concepts for coefficients and additions coming from integers, reals, rationals, fields, or even rigs. Under \mathbb{Z}_2 -additions, we have

$$c + c = \sum_{i=1}^k 0\sigma_i = 0.$$

The p -chains form a group under the addition '+' where the identity is the chain $0 = \sum_{i=1}^k 0\sigma_i$, and the inverse of a chain c is c itself since $c + c = 0$. This group, called the p -th chain group, is denoted $C_p := C_p(\mathcal{K})$.

12 Boundaries, cycles, homology

The chain groups at different dimensions are related by a boundary operator that, given a p -simplex, returns the $(p - 1)$ -chain of its boundary $(p - 1)$ -simplices. If $\sigma = \{v_0, \dots, v_p\}$ is a p -simplex, its boundary is

$$\partial_p \sigma = \sum_{i=0}^p \{v_0, \dots, \hat{v}_i, \dots, v_p\}$$

where \hat{v}_i indicates that the vertex v_i omitted. Extending ∂_p to a p -chain, we obtain a $(p - 1)$ -chain:

$$\partial_p : C_p \rightarrow C_{p-1}.$$

For example, for a 2-chain $\{a, b, c\} + \{b, c, d\}$, we get

$$\partial_2(\{a, b, c\} + \{b, c, d\}) = (\{a, b\} + \{b, c\} + \{c, a\}) + (\{b, c\} + \{c, d\} + \{d, a\}) = \{a, b\} + \{c, a\} + \{c, d\} + \{d, a\}.$$

It means that from the two triangles sharing the edge $\{b, c\}$, the boundary operator returns the four boundary edges that are not shared. Extending the boundary operator to the chains groups, we obtain the following sequence of homomorphisms for a simplicial complex with simplices of dimension at most $p \geq 0$:

$$C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} C_{p-2} \quad \cdots \quad C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} = \emptyset. \quad (1)$$

Fact 3.

1. C_p is a free, abelian group—it has a basis so that every element can be expressed uniquely as a sum of the elements in the basis. Commutativity of $+$ operation makes it abelian.
2. Each p -simplex is a basis element for C_p because any p -chain is a unique subset of the p -simplices. The rank of C_p is therefore n , the number of p -simplices.

Cycle and boundary groups.

Definition 34. A p -chain c is a p -cycle if $\partial c = 0$. In words, a chain that has empty boundary is a cycle. All p -cycles together form the p -cycle group $Z_p := Z_p(\mathcal{K})$ under the addition that is used to define the chain groups. In terms of the boundary operator, we have $\ker \partial_p = Z_p$.

For example, the 1-chain $\{a, b\} + \{b, c\} + \{c, a\}$ is a 1-cycle since

$$\partial(\{a, b\} + \{b, c\} + \{c, a\}) = (\{a\} + \{b\}) + (\{b\} + \{c\}) + (\{c\} + \{a\}) = 0.$$

Also, observe that the above 1-chain is the boundary of the triangle $\{a, b, c\}$. It's not accident that the boundary of a simplex is a cycle. In general, the boundary of a p -chain is a $(p - 1)$ -cycle. This is a very useful fact in homology theory.

Lemma 5. $\partial_{p-1} \partial_p c = 0$.

PROOF. It is sufficient to show that $\partial_{p-1}\partial_p\sigma = 0$ for a p -simplex σ . Observe that $\partial_p\sigma$ is the set of all $(p-1)$ -faces of σ and every $(p-2)$ -faces of σ is contained in exactly two $(p-1)$ -faces. Thus, $\partial_{p-1}(\partial_p\sigma) = 0$. \square

All $(p-1)$ -chains that can be obtained by applying the boundary operator ∂_p on p -chains form a subgroup of $(p-1)$ -chains, $B_{p-1} = \partial_p(C_p)$, or in other words, the image of the boundary homomorphism is a boundary group, $\text{Im}\partial_p = B_{p-1}$. We have $\partial_{p-1}B_{p-1} = 0$ due to Lemma 5 and hence $B_{p-1} \subseteq Z_{p-1}$.

Fact 4.

1. $B_p \subseteq Z_p \subseteq C_p$.
2. Both B_p and Z_p are also free and abelian since C_p is.

Homology groups. The homology groups classify the cycles in a cycle group by putting together those cycles in the same class that differ by a boundary. From group theoretic point of view, this is done by taking the quotient of the cycle groups with the boundary groups, which is allowed since the boundary group is a subgroup of the cycle group.

Definition 35. The p -th homology group is $H_p = Z_p/B_p$. Since we use a field, namely \mathbb{Z}_2 , to define the group operation, H_p is a vector space and its dimension is the Betti number,

$$\beta_p = \text{dimension } H_p.$$

Every element of H_p is obtained by adding a p -cycle $c \in Z_p$ to the entire boundary group, $c + B_p$, which is a coset of B_p in Z_p . All cycles constructed by adding an element of B_p to c_p form the class $[c]$, referred to as the *homology class* of c . Two cycles in the same homology class are called *homologous*. By definition $[c] = [c']$ if c and c' are homologous. Also, observe that the group operation for H_p is defined by $[c] + [c'] = [c + c']$.

Example. Consider the boundary complex \mathcal{K} of a tetrahedron which consists of four triangles, six edges, and four vertices. The 2-chain of four triangles make a 2-cycle c because its boundary is 0. Since \mathcal{K} does not have any 3-simplex (the tetrahedron is not part of the complex), this 2-cycle cannot be added to any 2-boundary other than 0 to form its class. Therefore, the homology class of c is c itself, $[c] = c$. There is no other 2-cycle in \mathcal{K} . Therefore, $H_2(\mathcal{K})$ is generated by c alone. Its dimension is only one. If the tetrahedron is included in the complex, c becomes a boundary element, and hence $[c] = [0]$. In that case, $H_2(\mathcal{K}) = 0$. Now, convince yourself that $H_1(\mathcal{K}) = 0$ no matter whether the tetrahedron belongs to \mathcal{K} or not.

Fact 5.

1. H_p is free and abelian (when defined over \mathbb{Z}_2).
2. It may not remain free when defined under \mathbb{Z} , the integer coefficients. In this case, there could be torsion subgroups.
3. The betti number, $\beta_p = \text{rank } H_p$, is given by $\beta_p = \text{rank } Z_p - \text{rank } B_p$.
4. There are exactly 2^{β_p} homology classes in H_p .

13 Induced homology

Continuous functions from a topological space to another topological space takes cycles to cycles and boundaries to boundaries. Therefore, they induce a map in their homology groups as well. We will restrict ourselves only to simplicial complexes. Simplicial maps are the counterpart of continuous maps between topological spaces.

Definition 36. A map $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ is called *simplicial* if for every simplex $\{v_0, \dots, v_k\} \in \mathcal{K}_1$, we have the simplex $\{f(v_0), \dots, f(v_k)\}$ in \mathcal{K}_2 .

Notice that a vertex map $f : V(\mathcal{K}_1) \rightarrow V(\mathcal{K}_2)$ does not necessarily extend to a simplicial map. But, every simplicial map is associated with a vertex map.

Fact 6. Every continuous map $f : |\mathcal{K}_1| \rightarrow |\mathcal{K}_2|$ can be approximated arbitrarily closely with a simplicial map on barycentric subdivisions of \mathcal{K}_1 and \mathcal{K}_2 .

Simplicial maps between simplicial complexes take cycles to cycles and boundaries to boundaries with the following definition.

Definition 37. Let $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be a simplicial map. The chain map $f_\# : C_p(\mathcal{K}_1) \rightarrow C_p(\mathcal{K}_2)$ corresponding to f is defined as follows. If $c = \sum \alpha_i \sigma_i$ is a p -chain, then $f_\#(c) = \sum \alpha_i \tau_i$ where

$$\tau_i = \begin{cases} f(\sigma_i), & \text{if } \sigma_i \text{ is a } p\text{-simplex in } \mathcal{K}_1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6. Let $\partial_{\mathcal{K}_1}$ and $\partial_{\mathcal{K}_2}$ denote the boundary maps (homomorphisms). Then, the induced chain maps commute with the boundary maps, that is, $f_\# \circ \partial_{\mathcal{K}_1} = \partial_{\mathcal{K}_2} \circ f_\#$.

The statement in the above proposition can also be represented with the following diagram, which we say commute since starting from the top left corner, one reaches to the same chain at the lower right corner using both paths—first going right and then down, or first going down and then right.

$$\begin{array}{ccc} C_p(\mathcal{K}_1) & \xrightarrow{f_\#} & C_p(\mathcal{K}_2) \\ \downarrow \partial_{\mathcal{K}_1} & & \downarrow \partial_{\mathcal{K}_2} \\ C_{p-1}(\mathcal{K}_1) & \xrightarrow{f_\#} & C_{p-1}(\mathcal{K}_2) \end{array}$$

By the commutativity of the above diagram, we see that $f_\#(Z_p(\mathcal{K}_1)) \subseteq Z_p(\mathcal{K}_2)$ and $f_\#(B_p(\mathcal{K}_1)) \subseteq B_p(\mathcal{K}_2)$. Therefore, we also have an induced map in the quotient space, namely,

$$f_*(Z_p(\mathcal{K}_1)/B_p(\mathcal{K}_1)) = f_\#(Z_p(\mathcal{K}_1))/f_\#(B_p(\mathcal{K}_1)).$$

Thus, we have an induced map in the homology groups

$$f_* : H_p(\mathcal{K}_1) \rightarrow H_p(\mathcal{K}_2).$$

A homology class $[c] = c + B_p$ in \mathcal{K}_1 is mapped to the homology class $f_\#(c) + f_\#(B_p)$ in \mathcal{K}_2 by f_* .