Lecture 5: Introduction to B-Spline Curves

The control in shape change is better achieved with B-spline curves than the Bézier curves. The degree of the curve is not dependent on the total number of points. B-splines are made out several curve segments that are joined “smoothly”. Each such curve segment is controlled by a couple of consecutive control points. Thus, a change in the position of a control point only propagates up to a predictable range.

B-spline Polynomial

Let \( p_0, \ldots, p_n \) be the control points. The nonrational form of a B-spline is given by

\[
p(u) = \sum_{i=0}^{n} p_i N_{i,D}(u)
\]

For Bézier curves the number of control points determine the degree of the basis functions. For a B-spline curve a number \( D \) determines its degree which is \( D - 1 \).

The basis functions are defined recursively as follows:

\[
N_{i,1}(u) = \begin{cases} 
1 & \text{if } t_i \leq u < t_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
N_{i,d}(u) = \frac{(u - t_i)N_{i,d-1}(u)}{t_{i+d-1} - t_i} + \frac{(t_{i+d} - u)N_{i+1,d-1}(u)}{t_{i+d} - t_{i+1}}
\]

If the denominator of any of the two terms on the RHS is zero, we take that term to be zero. The range of \( d \) is given by \( d : d = 2, \ldots, D \).

The \( t_j \) are called knot values, and a set of knots form a knot vector. If we have \( n \) control points and we want a B-spline curve of degree \( D - 1 \) we need \( T = n + D + 1 \) knots. If we impose the condition that the curve go through the end points of the control polygon, the knot values will be:

\[
t_j = \begin{cases} 
0 & \text{if } j < D \\
j - D + 1 & \text{if } D \leq j \leq n \\
n - D + 2 & \text{if } n < j \leq n + D
\end{cases}
\]

The knots range from 0 to \( n + D \), the index \( i \) of basis functions ranges from 0 to \( n \). So, there are always \( n + 1 \) basis functions each of which depends on \( D \) knots. The knot vector takes the form

\[
T = \{\alpha, \ldots, \alpha, t_D, \ldots, t_n, \beta, \ldots, \beta\}
\]

where the end knots \( \alpha \) and \( \beta \) are repeated with multiplicity \( D \). We can parameterize the entire curve over \([0,1]\). But, for simplicity we will use the interval \([\alpha = 0, \beta = n - D + 2]\). If we space the knots uniformly we get uniform B-splines. Otherwise, we get nonuniform B-splines as is done in the previous discussion.

\(^1\)Note by Tamal K. Dey
Basis functions

Let us assume that we have six control points \( n = 5 \). Then, \( N_{i,1} \) will require \( 5 + 1 + 1 = 7 \) knots \( \{0, 1, 2, 3, 4, 5, 6\} \). We obtain:

\[
N_{0,1}(u) = \begin{cases} 
1 & \text{if } 0 \leq u < 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
N_{1,1}(u) = \begin{cases} 
1 & \text{if } 1 \leq u < 2 \\
0 & \text{otherwise}
\end{cases}
\]

\[
N_{i,1}(u) = \begin{cases} 
1 & \text{if } i \leq u < i+1 \\
0 & \text{otherwise}
\end{cases}
\]

These are box functions and the corresponding \( B \)-spline is \( p(u) = p_i \) for \( i \leq u < i + 1 \), i.e., the control points itself.

Now let's consider the basis functions \( N_{i,2}(u) \). This will need \( N_{i,1} \)'s.

\[
N_{0,1}(u) = \begin{cases} 
1 & \text{if } u = 0 \\
0 & \text{otherwise}
\end{cases}
\]

\[
N_{1,1}(u) = \begin{cases} 
1 & \text{if } 0 \leq u < 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
N_{i,1}(u) = \begin{cases} 
1 & \text{if } 4 \leq u < 5 \\
0 & \text{otherwise}
\end{cases}
\]

This gives:

\[
N_{0,2}(u) = (1 - u)N_{1,1}(u)
\]

\[
N_{1,2}(u) = uN_{1,1}(u) + (2 - u)N_{2,1}(u)
\]

\[
N_{2,2}(u) = (u - 1)N_{2,1}(u) + (3 - u)N_{3,1}(u)
\]

\[
N_{3,2}(u) = (u - 2)N_{3,1}(u) + (4 - u)N_{4,1}(u)
\]

\[
N_{4,2}(u) = (u - 3)N_{4,1}(u) + (5 - u)N_{5,1}(u)
\]

\[
N_{5,2}(u) = (u - 4)N_{5,1}(u)
\]

We have the curve:

\[
p(u) = \begin{cases} 
(1 - u)p_0 + up_1 & 0 \leq u < 1 \\
(2 - u)p_1 + (u - 1)p_2 & 1 \leq u < 2 \\
(3 - u)p_2 + (u - 2)p_3 & 2 \leq u < 3 \\
(4 - u)p_3 + (u - 3)p_4 & 3 \leq u < 4 \\
(5 - u)p_4 + (u - 4)p_5 & 4 \leq u \leq 5
\end{cases}
\]

These are the line segments joining the control points.