Lecture 5: Introduction to B-Spline Curves [1]

The control in shape change is better achieved with B-spline curves than the Bézier curves. The degree of the curve is not dependent on the total number of points. B-splines are made out several curve segments that are joined "smoothly". Each such curve segment is controlled by a couple of consecutive control points. Thus, a change in the position of a control point only proapagates upto a predictable range.

B-spline Polynomial

Let $\mathbf{p}_0, ..., \mathbf{p}_n$ be the control points. The nonrational form of a B-spline is given by

$$\mathbf{p}(u) = \Sigma_{i=0}^{n} \mathbf{p}_i N_{i,D}(u)$$

For Bézier curves the number of control points determine the degree of the basis functions. For a B-spline curve a number D determines its degree which is $D-1$.

The basis functions are defined recursively as follows:

$$\begin{aligned} N_{i,1}(u) &= 1 \; if \; t_i \leq u < t_{i+1} \\ &= 0 \; otherwise \end{aligned}$$

and

$$N_{i,d}(u) = \frac{(u - t_i) N_{i,d-1}(u)}{t_{i+d-1} - t_i} + \frac{(t_{i+d} - u) N_{i+1,d-1}(u)}{t_{i+d} - t_{i+1}}$$

If the denominator of any of the two terms on the RHS is zero, we take that term to be zero. The range of d is given by $d : d = 2, .., d = D$.

The t_j are called *knot* values, and a set of knots form a *knot* vector. If we have n control points and we want a B-spline curve of degree $D-1$ we need $T = n + D + 1$ knots. If we impose the condition that the curve go through the end points of the control polygon, the knot values will be:

$$\begin{aligned} t_j &= 0 \; if \; j < D \\ t_j &= j - D + 1 \; if \; D \leq j \leq n \\ t_j &= n - D + 2 \; if \; n < j \leq n + D \end{aligned}$$

The knots range from 0 to $n + D$, the index i of basis functions ranges from 0 to n. So, there are always $n+1$ basis functions each of which depends on D knots. The knot vector takes the form

$$\mathbf{T} = \{\alpha, ..., \alpha, t_D, ..., t_n, \beta, ..., \beta\}$$

where the end knots α and β are repeated with multiplicity D. We can parameterize the entire curve over $[0, 1]$. But, for simplicity we will use the interval $[\alpha = 0, \beta = n - D + 2]$. If we space the knots uniformly we get uniform B-splines. Otherwise, we get nonuniform B-splines as is done in the previous discussion.

[1] Note by Tamal K. Dey

Basis functions

Let us assume that we have six control points ($n = 5$). Then, $N_{i,1}$ will require $5 + 1 + 1 = 7$ knots $\{0, 1, 2, 3, 4, 5, 6\}$. We obtain:

$$\begin{aligned}
N_{0,1}(u) &= 1 \; if \; 0 \leq u < 1 \\
&= 0 \; otherwise \\
N_{1,1}(u) &= 1 \; if \; 1 \leq u < 2 \\
&= 0 \; otherwise \\
&. \\
&. \\
N_{5,1}(u) &= 1 \; if \; 5 \leq u \leq 6 \\
&= 0 \; otherwise
\end{aligned}$$

These are box functions and the corresponding B-spline is $\mathbf{p}(u) = \mathbf{p}_i$ for $i \leq u < i + 1$, i.e., the control points itself.

Now lets consider the basis functions $N_{i,2}(u)$. This will need $N_{i,1}$'s.

$$\begin{aligned}
N_{0,1}(u) &= 1 \; if \; u = 0 \\
&= 0 \; otherwise \\
N_{1,1}(u) &= 1 \; if 0 \leq u < 1 \\
&= 0 \; otherwise \\
&. \\
&. \\
N_{5,1}(u) &= 1 \; if \; 4 \leq u \leq 5 \\
&= 0 \; otherwise
\end{aligned}$$

This gives:

$$\begin{aligned}
N_{0,2}(u) &= (1 - u)N_{1,1}(u) \\
N_{1,2}(u) &= uN_{1,1}(u) + (2 - u)N_{2,1}(u) \\
N_{2,2}(u) &= (u - 1)N_{2,1}(u) + (3 - u)N_{3,1}(u) \\
N_{3,2}(u) &= (u - 2)N_{3,1}(u) + (4 - u)N_{4,1}(u) \\
N_{4,2}(u) &= (u - 3)N_{4,1}(u) + (5 - u)N_{5,1}(u) \\
N_{5,2}(u) &= (u - 4)N_{5,1}(u)
\end{aligned}$$

We have the curve:

$$\begin{aligned}
\mathbf{p}(u) &= (1 - u)\mathbf{p}_0 + u\mathbf{p}_1 \;\; 0 \leq u < 1 \\
&= (2 - u)\mathbf{p}_1 + (u - 1)\mathbf{p}_2 \;\; 1 \leq u < 2 \\
&= (3 - u)\mathbf{p}_2 + (u - 2)\mathbf{p}_3 \;\; 2 \leq u < 3 \\
&= (4 - u)\mathbf{p}_3 + (u - 3)\mathbf{p}_4 \;\; 3 \leq u < 4 \\
&= (5 - u)\mathbf{p}_4 + (u - 4)\mathbf{p}_5 \;\; 4 \leq u \leq 5
\end{aligned}$$

These are the line segments joining the control points.