Lecture 2: Bézier Curves I ¹

Named after P. Bézier these parametric curves *approximates* a set of points called *control* points. Unlike Hermite interpolants these do not interpolate the points. The shape of the curve can be controled by moving the control points. But, this does not give the predictable control as the change in shape is global rather than local. We will see later that *B*-splines achieve this.

**Parametric form:**

Let \( p_0, ..., p_n \) be a set of \( n \) control points forming the vertices of a control polygon. The general equation is:

\[
p(u) = \sum_{i=0}^{n} p_i f_i(u), \quad u \in [0, 1]
\]

The basis functions \( f_i \)'s are chosen so that the following holds.

- The curve goes through \( p_0 \) and \( p_n \).
- The tangent at \( p_0 \) and \( p_n \) are given by \( p_1 - p_0 \) and \( p_n - p_{n-1} \).
- We can require higher order derivatives at the endpoints be controled by appropriate number of points.
- \( f_i(u) \) is symmetric with respect to \( u \) and \( 1 - u \).

**Basis functions:**

Bernstein polynomials \( B_{i,n} \) are chosen for basis functions \( f_i(u) = B_{i,n} \) where

\[
B_{i,n}(u) = \binom{n}{i} u^i (1 - u)^{n-i}
\]

These are \( n \)th degree polynomials.

\[
B_{0,2} = (1 - u)^2 \\
B_{1,2} = 2u(1 - u) \\
B_{2,2} = u^2
\]

So, with three control points we have

\[
p(u) = (1 - u)^2 p_0 + 2u(1 - u)p_1 + u^2 p_2
\]

Similarly, one can get cubic Bézier curve using four control points.

\[
p(u) = (1 - u)^3 p_0 + 3u(1 - u)^2 p_1 + 3u^2(1 - u)p_2 + u^3 p_3
\]

¹Note by Tamal K. Dey
Matrix forms:

The cubic curve can be written as:

\[ p(u) = [(1 - 3u + 3u^2 - u^3) (3u - 6u^2 + 3u^3) (3u^2 - 3u^3) u^3][p_0 \ p_1 \ p_2 \ p_3]^T \]

or

\[ p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}^T \]

Compactly, we can write \( p(u) = U M_b P \) where \( U \) is the \( u \)-vector, \( M_b \) is the co-efficient matrix and \( P \) is the point vector above. We will use this matrix form later in subdivisions.

Affine invariance:

Bézier curves are invariant under affine transformation, i.e., translation, rotation, scaling, or shear.

This means that the following should be true. Let

\[ p(u_i) = \sum_{i=0}^n p_i B_{i,n}(u_i) \]

Apply the affine transformation \( A \) so that \( p'(u_i) = A p(u_i) \). This point is same as the point obtained after transforming the control points and then using the approximation, i.e., \( p'(u_i) = \sum_{i=0}^n p_i B_{i,n}(u_i) \)

A Bézier curve is also invariant under affine reparameterization. So, if \( u \in [0,1] \) and \( v \in [a,b] \) then using \( u = \frac{(v-a)}{(b-a)} \) we have

\[ \sum_{i=0}^n p_i B_{i,n}(u) = \sum_{i=0}^n p_i B_{i,n} \left( \frac{v-a}{b-a} \right) . \]

Convex hull property

Observe that \( \sum_{i=0}^n B_{i,n}(u) = (u + (1-u))^n = 1 \) and also \( B_{i,n}(u) \geq 0 \) for \( u \in [0,1] \). Thus the points on \( p(u) \) is a convex combination of the control points, and thus reside inside their convex hull.