Lecture 17: Subdividing Biquadratic $B$-spline Surfaces

Matrix equation

We will generalize the matrix subdivision technique for $B$-spline curves to surfaces. Let us consider the biquadratic $B$-spline surface. The $P$ matrix in this case is a $3 \times 3$ array.

\[
p(u, w) = \begin{bmatrix} u^2 & u \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} P M^T W^T
\]

where

\[
M = \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}
\]

Subdividing a surface patch

A surface patch is generated with a control polyhedron of four rectangles spanning the $3 \times 3$ array of control points. With a subdivision scheme of splitting at $u = 1/2, v = 1/2$ we create four patches out of a single patch that will be defined by $4 \times 4$ array of new control points. We express the new control points in terms of the old ones using the matrix subdivision technique. We consider the subpatch $p(u, w)$ for $u \in [0, 1/2]$ and $w \in [0, 1/2]$. Define this surface patch as

\[
p'(u, w) = p\left(\frac{u}{2}, \frac{w}{2}\right)
\]

We have

\[
p'(u, w) = p\left(\frac{u}{2}, \frac{w}{2}\right)
\]

\[
= \begin{bmatrix} \frac{u^2}{4} & \frac{u}{2} \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} P M^T W^T
\]

\[
= U \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} W^T
\]

\[
= U M M^{-1} X M P M^T X^T (M^{-1})^T M^T W^T
\]

where

\[
P' = S P S^T
\]

and

\[
S = M^{-1} X M
\]

\[1\text{Note by Tamal K. Dey, Ohio State U.}\]
From the above we can conclude that \( p'(u, w) \) is generated with the new control points \( P' \). If we calculate the splitting matrix we obtain
\[
S = \frac{1}{4} \begin{bmatrix}
  3 & 1 & 0 \\
  1 & 3 & 0 \\
  0 & 3 & 1 \\
\end{bmatrix}
\]

Using \( P' = S P S^T \) we get the new control points as:
\[

p_{00}' = \frac{1}{16} (9p_{00} + 3p_{10} + 3p_{01} + p_{11}) \\
p_{01}' = \frac{1}{16} (3p_{00} + p_{10} + 9p_{01} + 3p_{11}) \\
p_{02}' = \frac{1}{16} (9p_{01} + 3p_{11} + 3p_{02} + p_{12}) \\
p_{10}' = \frac{1}{16} (3p_{00} + 9p_{10} + p_{01} + 3p_{11}) \\
p_{11}' = \frac{1}{16} (p_{00} + 3p_{10} + 3p_{01} + 9p_{11}) \\
p_{12}' = \frac{1}{16} (3p_{01} + 9p_{11} + p_{02} + 3p_{12}) \\
p_{20}' = \frac{1}{16} (9p_{10} + 3p_{20} + 3p_{11} + p_{21}) \\
p_{21}' = \frac{1}{16} (3p_{10} + p_{20} + 9p_{11} + 3p_{21}) \\
p_{22}' = \frac{1}{16} (9p_{11} + 3p_{21} + 3p_{12} + p_{22})
\]

Observations

If we observe the above equations for new control points, we see that four points of a rectangle in the original mesh generate a new point with the weights of \( 9 - 3 - 3 - 1 \). Let us call the new point \( v_F \) to denote that it is the new point corresponding to the vertex \( v \) in the rectangle \( F \) with weight \( 9 \). Then the weights of the two vertices adjacent to it will have weights \( 3 \). The opposite vertex to \( v \) in \( F \) has weight \( 1 \). Thus, each of the four vertices in \( F \) generates a new vertex \( v_F \).
Also, one can observe that a new point can be thought of as weighing an edge in the $3 - 1$ ratio and then weighing these new points again with $3 - 1$ ratio. This is just the generalization of Chaikin’s algorithm for quadratic $B$-spline curves.