Randomized Rounding

- Similar to the deterministic rounding, except that optimal fractional solution is rounded randomly according to some probability distribution.

- Goal is to obtain a good approximation ratio with high probability.

- Independently run the rounding multiple times to improve the above probability.
O(\log n) - approximation for Set Cover

Problem: Given sets $S_1, S_2, \ldots, S_m$ of \{1, 2, \ldots, n\}, find smallest subset $C$ s.t. $C \cap S_k \neq \emptyset$ for each $k$.

LP relaxation:

$$\min \sum_i x_i$$
$$\sum_{i \in S_k} x_i \geq 1 + k$$
$$0 \leq x_i \leq 1.$$  

Randomized Rounding:

$$y_i = \text{rounded}(x_i)$$
$$y_i = 1 \text{ with prob. } (x_i)$$
$$y_i = 0 \text{ with prob. } (1-x_i).$$
\[ \forall K, \ Pr[ S_K \text{ gets covered}] = 1 - (1-x_{k_1}) \cdots (1-x_{k_K}) \]
when \( S_K = \{ k_1, k_2, \ldots, k_K \} \).

We have
\[ (1-x_{k_1}) \cdots (1-x_{k_K}) \leq \left( \frac{e - (x_{k_1} + x_{k_2} + \cdots + x_{k_K})}{e} \right)^K \leq (1-\frac{1}{e})^K \leq \frac{1}{e} \]

So,
\[ Pr[ S_K \text{ gets covered}] \geq 1 - \frac{1}{e} \]

- Now repeat the randomized rounding \( t \) times and take the union of all the sets produced.

- This means we round \( x_i \) to 0 with probability \( (1-x_i)^t \), that \( x_i \) is rounded to 0 if it is rounded to 0 in all \( t \) rounds.
Prob. \[ s_k \text{ remains uncovered} \leq \left( \frac{1}{e} \right)^t \text{ after } t \text{ rounds.} \]

\[
\text{Prob.} [\text{any of } s_k \text{ remains uncovered}] \\
\leq m \cdot \left( \frac{1}{e} \right)^t .
\]

If we take \( t = \log_e m + 1 \), we see

\[
\text{Prob.} [\text{any of } s_k \text{ remains uncovered}] \\
\leq \frac{1}{e^{m-1}} m = \frac{1}{e} < 1
\]

Size bound

\[
E[\text{size of } C \text{ after one round}] \\
= \sum_i x_i \\
= V(\text{LP}) \leq \text{OPT} \\
E[\text{size of } C \text{ after } t \text{ rounds}] \\
\leq t \cdot \text{OPT}
\]
By Markov's inequality,

\[
\text{Prob.}[\text{size of } C > P.\text{OPT}] \\
\leq \frac{E[\text{size of } C]}{P.\text{OPT}} \\
\leq \frac{t.\text{OPT}}{P.\text{OPT}} = \frac{t}{P}
\]

\[
\text{Prob.}[\text{all } Sk \text{ covered and size } C \leq P.\text{OPT}] \\
\geq (1 - m(\frac{1}{e})^t)(1 - \frac{t}{P}) \\
\geq 1 - m(\frac{1}{e})^t - \frac{t}{P}
\]

Put \( t = \Theta(\log m) \) and \( P = 4t \) so that \( 1 - m(\frac{1}{e})^t - \frac{t}{P} \geq \frac{1}{2} \).

**Thm.** This algorithm produces a solution with an \( O(\log m) \) approximation factor with probability at least \( \frac{1}{2} \).