- Ford-Fulkerson's algorithm takes $O(\text{If} \times \text{IE})$ time which could be costly.

- The choice of paths in the residual network makes a difference.

- Edmond-Karp algorithm chooses the augmenting path by a breadth-first search. This means the augmenting path $p: s \rightarrow t$ is the shortest path in $G_f$ w.r.t. # of edges (property of BFS).
We analyze that E-K algorithm takes $O(VE^2)$ time.

**Lemma 1** For all vertices $v \in V - \{s, t\}$, the shortest distance $d_f(s, v)$ in $G_f$ increases monotonically with flow augmentation.

**Proof.** Suppose $d_f(s, v)$ decreases, i.e., $d_{f'}(s, v) < d_f(s, v)$ where $f'$ is the flow just after $f$.

* Assume $u$ be the vertex with min. $d_{f'}(s, u)$ for which the decrease occurs.

* Let $P = s \rightarrow u \rightarrow v$ be the S.P. in $G_{f'}$.

1. $d_{f'}(s, u) = d_f(s, u) - 1$ and $(u, v) \in E_{f'}$.

* Because of our choice of $u$,

2. $d_{f'}(s, u) > d_f(s, u)$.

**Claim:** $(u, u) \notin E_{f'}$.

If $(u, u)$ were in $E_f$, we would have

$$d_f(s, u) \leq d_f(s, u) + 1 \leq d_{f'}(s, u) + 1 = d_{f'}(s, u),$$

contradicting.*
• We have $(u, v) \notin E_f$ and $(u, v) \in E_f'$
  \[ \Rightarrow \text{augmentation increased flow along from} \]
  \[ \text{u to v. E-K algorithm augments only along shortest path} \]
  \[ \Rightarrow \text{s.p. in } G_f \text{ from } s \text{ to } v \text{ has } (u, v) \text{ as} \]
  \[ \text{the last edge} \]

• \[ \delta_f(s, u) = \delta_f(s, u) - 1 \]
  \[ \leq \delta_{f'}(s, u) - 1 \quad \color{red}{(\text{2})} \]
  \[ = \delta_{f'}(s, u) - 2 \quad \color{red}{(\text{1})} \]
  \[ \text{Contradicts again } \ast. \]

• Conclusion: vertex such as \( u \)
  \[ \text{does not exist.} \]

**Theorem:** Total # of flow augmentations in E-K algorithm is \( O(VE) \).

**Proof.** \((u, v) \text{ is } \underline{\text{critical}} \text{ in } G_f \text{ if } (u, v) \in \mathcal{P} \)

s.t. \[ \mathcal{C}_f(P) = \mathcal{C}_f(u, v). \]
- prove that each edge can become critical only at most \( \lfloor \sqrt{1/2-n} \rfloor \) times, which also bounds \# augmentations.

- consider \((u, v)\) being critical.
  
  \[ \delta_f(s, u) = \delta_f(s, v) + 1 \]

- Edge \((u, v)\) disappears from \( G_f \).

- \((u, u)\) can reappear only if \((v, u)\) appears on augmenting path later for a flow \( f' \).

- \[ \delta_{f'}(s, u) = \delta_{f'}(s, v) + 1 \]
  \[ \geq \delta_f(s, u) + 1 \]
  \[ \text{(Lemma)} \]
  \[ = \delta_f(s, u) + 2 \]

- distance of \( u \) from \( s \) increases by at least 2 for two consecutive times when \((u, u)\) becomes critical.

- Since distance cannot be more than \( O(\sqrt{n}) \), \((u, u)\) can be critical only \( O(\sqrt{n}) \) times.

- So, \( |E| \) edges can become \( O(\sqrt{|V|}) \) times critical bounding \# augmentations.