Artificial Intelligence

Intro to Probability

Uncertainty and Vagueness

• Life is not always exact and certain
  – Do the best with what we do know
  – People draw conclusions when available information is uncertain, vague, or incomplete

• Sources of uncertainty
  – **Unreliable data**
    • Defective measurement device
  – **Incomplete data**
    • Only partial data available
  – **Imprecise data/rules**
    • Approximations of data
    • Rules for drawing conclusions may also be imprecise
Handling Uncertain Knowledge

• Consider dental diagnosis using first-order logic

\[ \forall p \ Symptom(p, \text{Toothache}) \rightarrow Disease(p, \text{Cavity}) \]

– Rule is actually wrong!
  • Not all patients with toothache have cavities
  • Need to add an almost unlimited list of possible causes

\[ \forall p \ Symptom(p, \text{Toothache}) \rightarrow Disease(p, \text{Cavity}) \]
\[ \lor \text{Disease}(p, \text{GumDisease}) \]
\[ \lor \text{Disease}(p, \text{ImpactedWisdom}) \]


Handling Uncertain Knowledge

• Use of first-order logic fails in diagnosis for three reasons
  – Laziness
    • Too much work to list all possible consequents
  – Theoretical ignorance
    • No complete theory for domain available
  – Practical ignorance
    • All tests have not been run to get data
Handling Uncertain Knowledge

- System’s knowledge can only provide a degree of belief about sentences
  - Sentences are in fact either true or false
- Probability theory
  - Deals with uncertainty
  - Assigns numerical degree of belief between 0-1
    - Probability of 0 is absolute belief that sentence is FALSE
    - Probability of 1 is absolute belief that sentence is TRUE
    - Probability of 0.8 (80%) that patient has cavity if has toothache
  - Assignment of probability depends on percepts received to date (evidence)

Basics of Probability

- Prior and Joint probabilities
- Conditional probabilities
- Bayes rule
Prior Probability

• Probability assessment before any evidence obtained
  – “Probability of event A” $\rightarrow P(A)$
  – e.g., $P(\text{Cavity}) = 0.1$
    • In the absence of any other information, 10% chance of the patient having a cavity

• Random variable assignments of exclusive values
  \[
  \begin{align*}
  P(\text{weather} = \text{Sunny}) &= 0.7 \\
  P(\text{weather} = \text{Rainy}) &= 0.2 \\
  P(\text{weather} = \text{Cloudy}) &= 0.08 \\
  P(\text{weather} = \text{Snow}) &= 0.02
  \end{align*}
  \]
  \[
  \text{weather} \rightarrow \{\text{Sunny, Rainy, Cloudy, Snow}\}
  \]
  Must add up to 1
Prior Probability

• Properties defining probability $P(A)$
  
  $0 \leq P(A) \leq 1$

  $P("certain\ event") = 1$

  $P(\neg A) = 1 - P(A)$

  $P(A) + P(\neg A) = 1$

• Simple die examples
  
  – Each side of die labeled from 1 to 6
  
  – Let event $A$ be a die stops with 1 showing on top: $P(A) = 1/6$

  – Let event $\neg A$ be a die stops with numbers other than a 1 showing on top

  \[ P(\neg A) = 5/6 \rightarrow P(\neg A) = 1 - P(A) = 1 - 1/6 = 5/6 \]

Prior Joint Probability

• Joint probability ("and")

  \[ P(Convex \wedge \neg Insured) = 0.06 \]

• Probability axiom of a disjunction ("or")

  \[ P(A \lor B) = P(A) + P(B) - P(A \land B) \]
Conditional Probability

• Probability assessment after evidence is obtained ("posterior probability")
  – "Probability of event A given all know is B" $\rightarrow P(A \mid B)$
  – e.g., $P(\text{Cavity} \mid \text{Toothache}) = 0.8$
    • If observe patient with toothache, 80% chance patient has cavity

• Conditional probability in terms of unconditionals
  $P(A \land B) = P(A \mid B)P(B)$ [Product rule]
  thus
  $$P(A \mid B) = \frac{P(A \land B)}{P(B)}$$

Note: If B conditionally (marginally) independent of A
  $P(A \land B) = P(A \mid B)P(B) = P(A)P(B)$
Conditional Probability

• As soon as get additional new information \( C \), must compute new conditional

\[
P(A | B \land C)
\]

• If \( A \) and \( B \) are **conditionally independent** of evidence \( C \)

\[
P(A | B \land C) = P(A | B)
\]

*Conditions independence is crucial to making probabilistic systems work when updating with several new pieces of evidence*

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Inference from Joint Probabilities

• Table of 2 propositions: *Cavity* and *Toothache*

<table>
<thead>
<tr>
<th></th>
<th>Toothache</th>
<th>¬Toothache</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cavity</td>
<td>0.04</td>
<td>0.06</td>
</tr>
<tr>
<td>¬Cavity</td>
<td>0.01</td>
<td>0.89</td>
</tr>
</tbody>
</table>

\[
P(Cavity \lor Toothache) = P(Cavity) + P(Toothache) - P(C \land T)
\]

\[
= (0.04 + 0.06) + (0.04 + 0.01) - 0.04 = 0.11
\]

\[
P(Cavity | Toothache) = P(Cavity \land Toothache)/P(Toothache)
\]

\[
= 0.04 / (0.04 + 0.01) = 0.80
\]
Bayes Rule

• Typical problem is to evaluate a hypothesis $H$ given evidence/data $E$: $P(H \mid E)$
• Recall our earlier product rule for a conditionals
  $P(H \mid E) = P(H \land E) / P(E)$
  But can be hard to get or understand $P(H \land E)$
• Recall product rule
  $P(A \land B) = P(A \mid B)P(B)$
  $P(B \land A) = P(B \mid A)P(A)$
  equate right-hand sides [as $P(A \land B) = P(B \land A)$]
  $P(A \mid B)P(B) = P(B \mid A)P(A)$
  $P(A \mid B) = P(B \mid A)P(A) / P(B)$
  $P(H \mid E) = P(E \mid H)P(H) / P(E)$
  \[\text{Bayes’ rule}\]
Bayes Rule

• Other variations of denominator

\[
P(H \mid E) = \frac{P(E \mid H)P(H)}{P(E)} = \frac{P(E \mid H)P(H) + P(E \mid \neg H)P(\neg H)}{\sum_i P(E \mid H_i)P(H_i)}
\]

"sum/integrate over all possible hypotheses"

\[
P(H \mid E) = \frac{P(E \mid H)P(H)}{\sum_i P(E \mid H_i)P(H_i)}
\]

Summary

• Decisions not always exact and certain
  – People draw conclusions when available information is uncertain, vague, or incomplete

• Probability laws
  – Prior probability
    • *A priori* initial information, independent of experience
    • Joint probability distribution
  – Conditional probability
    • Information based on additional evidence
  – Bayes rule
    • Updates belief measures in response to evidence