Artificial Intelligence

First-order logic

Previously

• Propositional logic
  – Simplest language
  – Its world only consists of facts (and “explicit rules”)
• Too puny of a language to represent knowledge of complex environments with many objects in a concise way
  – Difficult to represent even the Wumpus world
    \[ B_{1,1} \Rightarrow P_{1,2} \lor P_{2,1} \]
    Would instead like to say, “squares adjacent to pits are breezy” (not enumerate for all possible squares!)
First-Order Logic

• Also called **first-order predicate calculus**
  – FOL, FOPC
• Makes stronger commitments
  – World consists of **objects**
  – Objects have **properties/relations** that distinguish them from other objects

Syntax of FOL: Basic Elements

• Constant symbols for specific objects
  \( KingJohn, 2, OSU, \ldots \)
• **Predicate** relations (True/False relations)
  \( Brother, Married, >, \ldots \)
• **Functions** (only one “value” for a given “input”)
  \( Sqrt(), LeftLegOf(), FatherOf(), \ldots \)
• **Variables**
  \( x, y, a, b, \ldots \)
• Connectives
  \( \wedge, \vee, \neg, \rightarrow, \leftrightarrow \)
• **Equality**
  \( = \)
• **Quantifiers**
  \( \forall, \exists \)
Atomic Sentences

• Collection of terms and relation(s) that together state facts

• Atomic sentence
  – \textit{predicate}(\textit{term}_1, ..., \textit{term}_n)
  – Or \textit{term}_i = \textit{term}_n

• Examples
  \textit{Brother}(\textit{Richard}, \textit{John})
  \textit{Married}(\textit{FatherOf}(\textit{Richard}), \textit{MotherOf}(\textit{John}))

Complex Sentences

• Made from atomic sentences using \textit{logical} connectives
  \textit{¬S}, \textit{S}_1 \land \textit{S}_2, \textit{S}_1 \lor \textit{S}_2, \textit{S}_1 \Rightarrow \textit{S}_2, \textit{S}_1 \Leftrightarrow \textit{S}_2

Example:
\textit{Older}(\textit{John}, 30) \Rightarrow \textit{¬Younger}(\textit{John}, 30)
Quantifiers

- Currently have logic that allows objects
- Now want to express properties of entire collections of objects
  - Rather than enumerate the objects by name
- Two standard quantifiers
  - Universal $\forall$
  - Existential $\exists$

Universal Qualification

- “For all …” (typically use implication $\Rightarrow$)
  - Allows for “rules” to be constructed
- $\forall <\text{variables}> <\text{sentence}>
  - Everyone at OSU is smart
    $\forall x ((\text{Person}(x) \land \text{At}(x, \text{OSU})) \Rightarrow \text{Smart}(x))$
- $\forall x P(x)$ is equivalent to conjunction of all instantiations of $P$
  $((\text{Person}(\text{John}) \land \text{At}(\text{John}, \text{OSU})) \Rightarrow \text{Smart}(\text{John}))$
  $\land ((\text{Person}(\text{Bob}) \land \text{At}(\text{Bob}, \text{OSU})) \Rightarrow \text{Smart}(\text{Bob}))$
  $\land ((\text{Person}(\text{Mary}) \land \text{At}(\text{Mary}, \text{OSU})) \Rightarrow \text{Smart}(\text{Mary}))$
  $\land \ldots$
Existential Quantification

- **“There exists …”** (typically use conjunction $\land$)
  - Makes a statement about some object (not all)
- $\exists $ variables $<$sentences$>$
  - Someone at OSU is smart
    $\exists x \ Person(x) \land At(x, \ OSU) \land Smart(x)$
- $\exists x \ P(x)$ is equivalent to disjunction of all instantiations of $P$
  - $(Person(John) \land At(John, \ OSU) \land Smart(John))$
  - $(Person(Bob) \land At(Bob, \ OSU) \land Smart(Bob))$
  - $(Person(Mary) \land At(Mary, \ OSU) \land Smart(Mary))$ ...
- Uniqueness quantifier
  - $\exists ! x$ says a unique object exists

Properties of Quantifiers

- Quantifier duality: Each can be expressed using the other

\[
\forall x \ Person(x) \implies Likes(x, \ IceCream) \quad \text{“Everybody likes ice cream”}
\]
\[
\neg \exists x \ Person(x) \land \neg Likes(x, \ IceCream) \quad \text{“Not exist anyone who does not like ice cream”}
\]
\[
\exists x \ Person(x) \land Likes(x, \ Broccoli) \quad \text{“Someone likes broccoli”}
\]
\[
\neg \forall x \ Person(x) \implies \neg Likes(x, \ Broccoli) \quad \text{“Not the case that everyone does not like broccoli”}
\]
Properties of Quantifiers

• Important relations

\[
\exists x \ P(x) = \neg \forall x \ \neg P(x) \\
\forall x \ P(x) = \neg \exists x \ \neg P(x)
\]

(A) \( P(x) \Rightarrow Q(x) \) is same as \( \neg P(x) \lor Q(x) \)

(B) \( \neg(P(x) \land Q(x)) \) is same as \( \neg P(x) \lor \neg Q(x) \)

(C) \( \neg(P(x) \lor Q(x)) \) is same as \( \neg P(x) \land \neg Q(x) \)

Proof of (A)

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\[ P(x) \Rightarrow Q(x) \]

is same as

\[ \neg P(x) \lor Q(x) \]
### Proof of (B)

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$\neg(P(x) \land Q(x))$ is same as $\neg P(x) \lor \neg Q(x)$

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### Proof of (C)

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$\neg(P(x) \lor Q(x))$ is same as $\neg P(x) \land \neg Q(x)$

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Conversion Example

1. \( \forall x \ Person(x) \Rightarrow Likes(x, IceCream) \)

   [use: \( \forall x P(x) = \neg \exists x \neg P(x) \)]

2. \( \neg \exists x \neg (Person(x) \Rightarrow Likes(x, IceCream)) \)

   [use: \( P(x) \Rightarrow Q(x) \) is same as \( \neg P(x) \lor Q(x) \)]

3. \( \neg \exists x \neg (\neg Person(x) \lor Likes(x, IceCream)) \)

   [distribute negatives]

4. \( \neg \exists x \ Person(x) \land \neg Likes(x, IceCream) \)

Nested Quantifiers

- \( \forall x \forall y \) is same as \( \forall y \forall x \)
- \( \exists x \exists y \) is same as \( \exists y \exists x \)
- \( \exists x \forall y \) is not same as \( \forall y \exists x \)
  \[ \exists y Person(y) \land (\forall x Person(x) \Rightarrow Loves(x,y)) \]
  “There is someone who is loved by everyone”

\[ \forall x Person(x) \Rightarrow \exists y Person(y) \land Loves(x,y) \]
“Everybody loves somebody”
(not guaranteed to be the same person)
Equality

- Equality symbol (=)
  - Make statements to the effect that two terms refer to the same object

“Henry is the Father of John”
\[ \text{Father}(John) = \text{Henry} \]

“Spot has at least two sisters”
\[ \exists x, y \quad \text{Sister}(x, \text{Spot}) \land \text{Sister}(y, \text{Spot}) \land \neg(x = y) \]

Kinds of Rules

- For “Squares are breezy near a pit”
  - **Diagnostic** rule
    - Lead from observed effects to hidden causes
      - “Infer cause from effect”
      \[ \forall y \quad \text{Breezy}(y) \Rightarrow \exists x \quad \text{Pit}(x) \land \text{Adjacent}(x, y) \]
  - **Causal** “model-based” rule
    - Hidden world properties causes certain percepts
      - “Infer effect from cause”
      \[ \forall x, y \quad \text{Pit}(x) \land \text{Adjacent}(x, y) \Rightarrow \text{Breezy}(y) \]
Summary

• First-order logic
  – Increased expressive power over Propositional Logic
  – Objects and relations are semantic primitives
  – Syntax: constants, functions, predicates, equality, quantifiers
    • Two standard quantifiers
      – Universal
      – Existential