Introduction to Algorithms
Graph Algorithms

CSE 680
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Bipartiteness

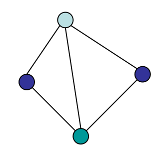
Graph $G = (V,E)$ is **bipartite** iff it can be partitioned into two sets of nodes A and B such that each edge has one end in A and the other end in B

Alternatively:

- Graph $G = (V,E)$ is bipartite iff all its cycles have even length
- Graph $G = (V,E)$ is bipartite iff nodes can be coloured using two colours

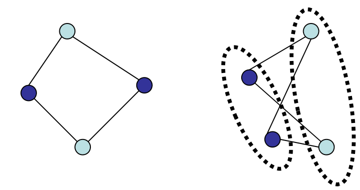
Question: given a graph G, how to test if the graph is bipartite?

Note: graphs without cycles (trees) are bipartite



non bipartite

bipartite:



Testing bipartiteness

Method: use BFS search tree

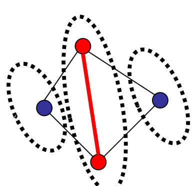
Recall: BFS is a rooted **spanning tree.**

Algorithm:

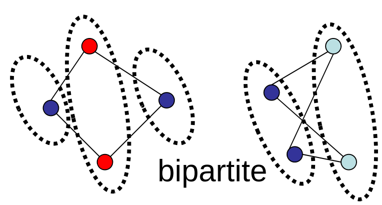
- Run BFS search and colour all nodes in odd layers red, others blue
- Go through all edges in adjacency list and check if each of them has two different colours at its ends - if so then G is bipartite, otherwise it is not

We use the following alternative definitions in the analysis:

- Graph $G = (V,E)$ is bipartite iff all its cycles have even length, or
- Graph $G = (V,E)$ is bipartite iff it has no odd cycle



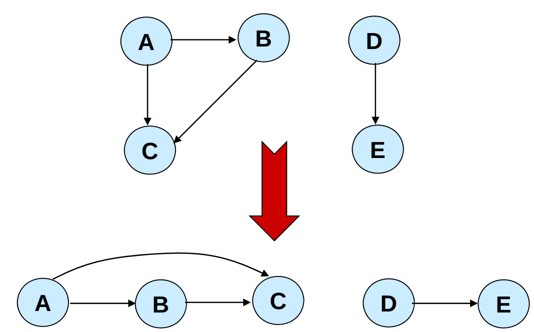
non bipartite



bipartite

Topological Sort

Want to "sort" or *linearize* a directed acyclic graph (DAG).



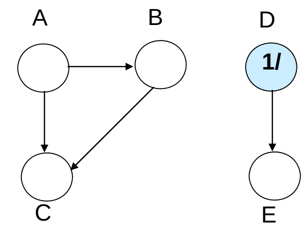
Topological Sort

- Performed on a DAG.
- Linear ordering of the vertices of G such that if $(u, v) \in E$, then u appears before v.

Topological-Sort (G)
1. call DFS(G) to compute finishing times $f[v]$ for all $v \in V$
2. as each vertex is finished, insert it onto the front of a linked list
3. **return** the linked list of vertices

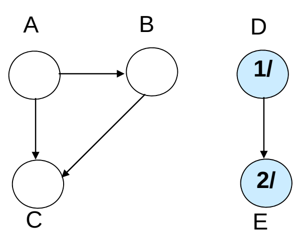
Time: $\Theta(V + E)$.

Example



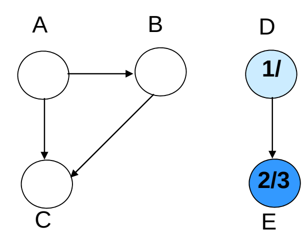
Linked List:

Example



Linked List:

Example



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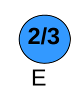
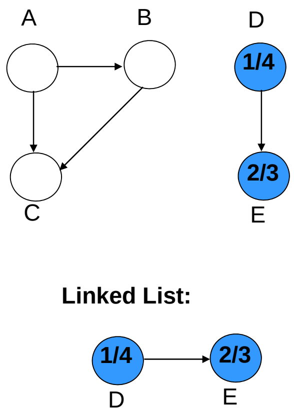
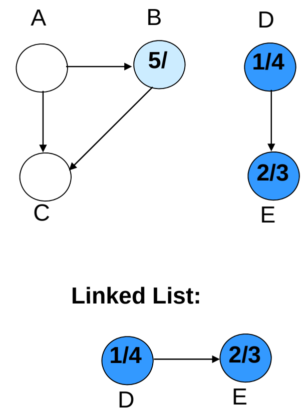


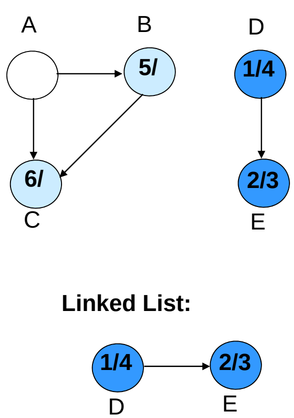
Example



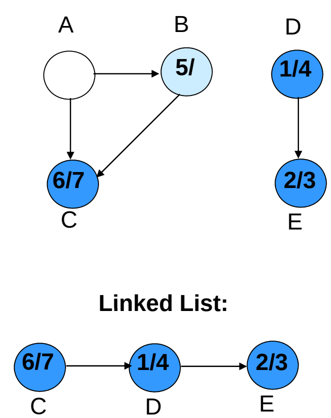
Example



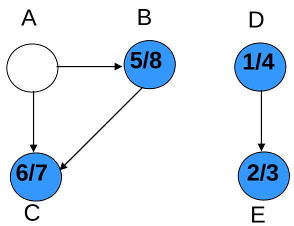
Example



Example



Example



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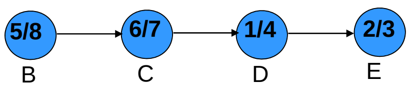

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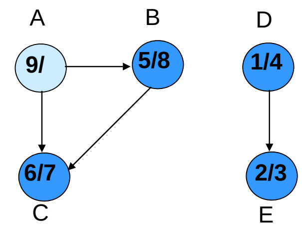

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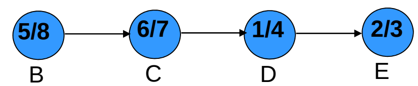
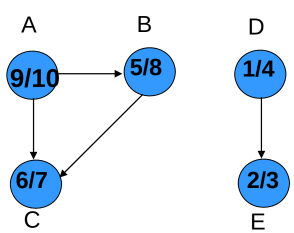


Example



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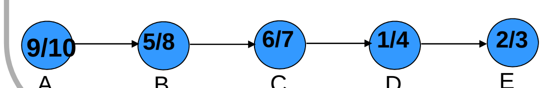
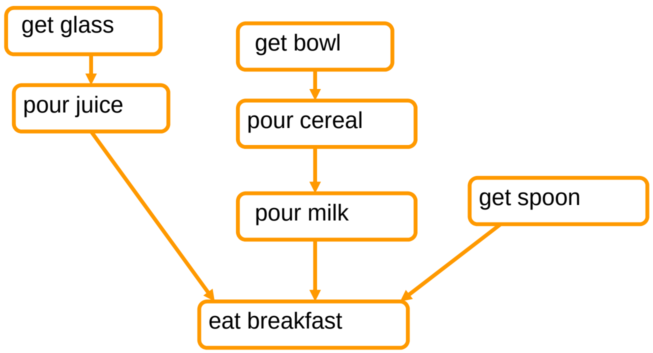


Precedence Example

- Tasks that have to be done to eat breakfast:
  - get glass, pour juice, get bowl, pour cereal, pour milk, get spoon, eat.
- Certain events must happen in a certain order (ex: get bowl before pouring milk)
- For other events, it doesn't matter (ex: get bowl and get spoon)

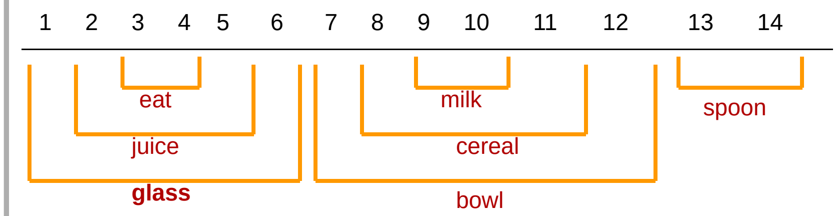
Precedence Example



Order: glass, juice, bowl, cereal, milk, spoon, eat.

Precedence Example

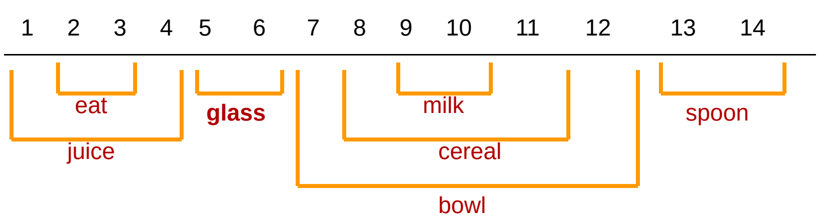
- Topological Sort



consider reverse order of finishing times:
spoon, bowl, cereal, milk, glass, juice, eat

Precedence Example

- What if we started with *juice?*



consider reverse order of finishing times:
spoon, bowl, cereal, milk, glass, juice, eat

Correctness Proof

- Show if $(u, v) \in E$, then $f[v] < f[u]$.
- When we explore (u, v), *what are their colors?*
  - Note, u is gray – we are exploring it
  - Is v gray?
    - *No*, because then v would be an ancestor of u.
    - $\Rightarrow (u, v)$ is a back edge.
    - $\Rightarrow$ a cycle (dag has no back edges).
  - Is v white?
    - Then v becomes descendant of u.
    - By parenthesis theorem, $d[u] < d[v] < f[v] < f[u]$.
  - Is v black?
    - Then v is already finished.
    - Since we're exploring (u, v), we have not yet finished u.
    - Therefore, $f[v] < f[u]$.

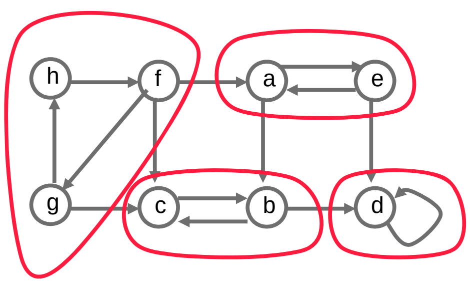
Strongly Connected Components

- Consider a directed graph.
- A strongly connected component (SCC) of the graph is a maximal set of nodes with a (directed) path between every pair of nodes.
  - If a path from *u* to *v* exists in the SCC, then a path from *v* to *u* also exists.
- Problem: Find all the SCCs of the graph.

Uses of SCC's

- Packaging software modules
  - Construct directed graph of which modules call which other modules
  - A SCC is a set of mutually interacting modules
  - Pack together those in the same SCC

SCC Example



four SCCs

Main Idea of SCC Algorithm

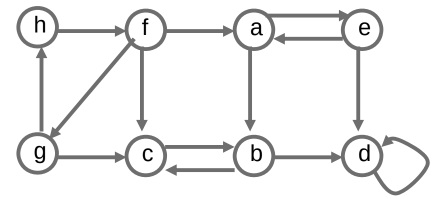
- DFS tells us which nodes are reachable from the roots of the individual trees
- Also need information in the other direction: is the root reachable from its descendants?
- Run DFS again on the ***transpose*** graph (reverse the directions of the edges)

SCC Algorithm

Input: directed graph G = (V,E)

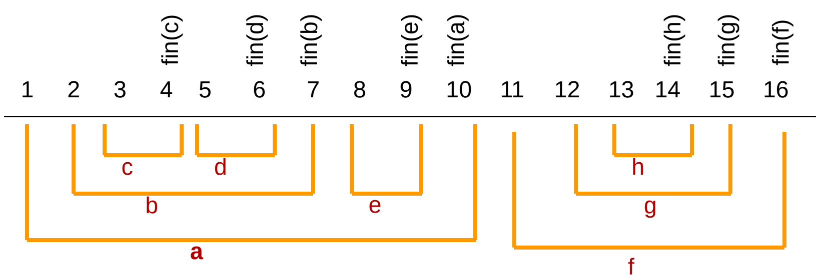
1. call DFS(G) to compute finishing times
2. compute G^T // transpose graph
3. call DFS(G^T), considering nodes in decreasing order of finishing times
4. each tree from Step 3 is a separate SCC of G

SCC Algorithm Example



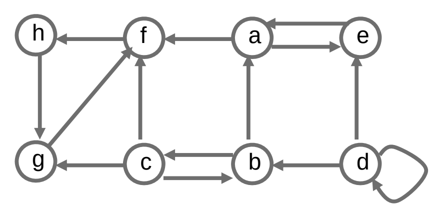
input graph - run DFS

After Step 1



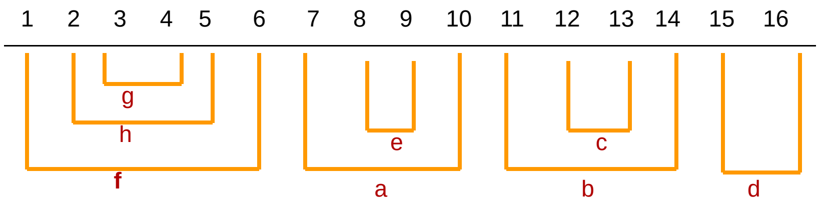
Order of nodes for Step 3: f, g, h, a, e, b, d, c

After Step 2



transposed input graph - run DFS with specified order of nodes

After Step 3



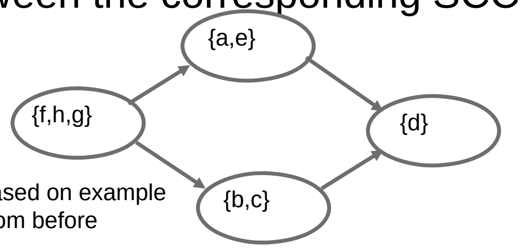
SCCs are {f,h,g} and {a,e} and {b,c} and {d}.

Run Time of SCC Algorithm

- Step 1: O(V+E) to run DFS
- Step 2: O(V+E) to construct transpose graph, assuming adjacency list rep.
  - Adjacency matrix is O(1) time w/ wrapper.
- Step 3: O(V+E) to run DFS again
- Step 4: O(V) to output result
- Total: O(V+E)

Component Graph

- $G^{SCC} = (V^{SCC}, E^{SCC})$.
- V^{SCC} has one vertex for each SCC in G.
- E^{SCC} has an edge if there's an edge between the corresponding SCC's in G.



G^{SCC} based on example graph from before

Component Graph Facts

- Claim: G^{SCC} is a directed acyclic graph.
  - Suppose there is a cycle in G^{SCC} such that component C_i is reachable from component C_j and vice versa.
  - Then C_i and C_j would not be separate SCCs.
- Lemma: If there is an edge in G^{SCC} from component C' to component C, then

  $$f(C') > f(C).$$

  - Consider any component C during Step 1 (running DFS on G)
  - Let d(C) be *earliest* discovery time of any node in C
  - Let f(C) be *latest* finishing time of any node in C