Introduction to Algorithms
Graph Algorithms

CSE 680
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Graphs

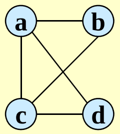
- *Graph G* = (*V*, *E*)
  - » *V* = set of vertices
  - » *E* = set of edges $\subseteq (V \times V)$
- Types of graphs
  - » Undirected: edge (*u*, *v*) = (*v*, *u*); for all *v*, (*v*, *v*) $\notin$ *E* (No self loops.)
  - » Directed: (*u*, *v*) is edge from *u* to *v*, denoted as $u \rightarrow v$. Self loops are allowed.
  - » Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbf{R}$.
  - » Dense: $|E| \approx |V|^2$.
  - » Sparse: $|E| << |V|^2$.
- $|E| = O(|V|^2)$

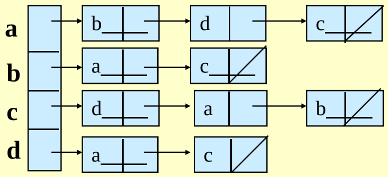
Graphs

- If $(u, v) \in E$, then vertex *v* is adjacent to vertex *u*.
- Adjacency relationship is:
  - » Symmetric if *G* is undirected.
  - » Not necessarily so if *G* is directed.
- If *G* is connected:
  - » There is a path between every pair of vertices.
  - » $|E| \geq |V| - 1$.
  - » Furthermore, if $|E| = |V| - 1$, then *G* is a tree.
- Other definitions in Appendix B (B.4 and B.5) as needed.

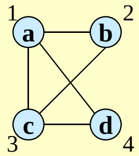
Representation of Graphs

- Two standard ways.
  - » Adjacency Lists.





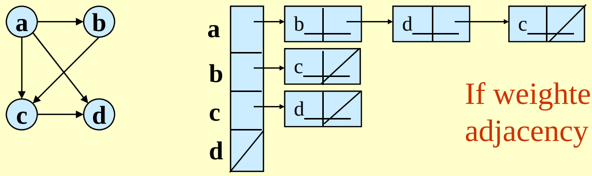
  - » Adjacency Matrix.



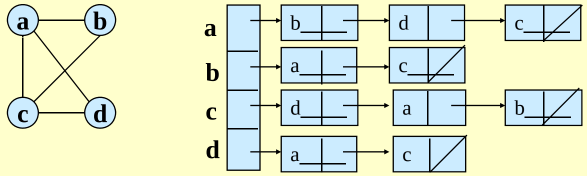
	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

Adjacency Lists

- Consists of an array *Adj* of |*V*| lists.
- One list per vertex.
- For $u \in V$, *Adj*[*u*] consists of all vertices adjacent to *u*.



If weighted, store weights also in adjacency lists.



Storage Requirement

- For directed graphs:
  - » Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{out-degree}(v) = |E|$$

No. of edges leaving *v*

  - » Total storage: $\Theta(|V| + |E|)$
- For undirected graphs:
  - » Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

No. of edges incident on *v*. Edge (*u*,*v*) is incident on vertices *u* and *v*.

  - » Total storage: $\Theta(|V| + |E|)$

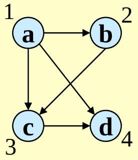
Pros and Cons: adj list

- Pros
  - » Space-efficient, when a graph is sparse.
  - » Can be modified to support many graph variants.
- Cons
  - » Determining if an edge $(u, v) \in G$ is not efficient.
    - Have to search in *u*'s adjacency list. Θ(degree(*u*)) time.
    - Θ(*V*) in the worst case.

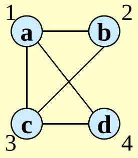
Adjacency Matrix

- $|V| \times |V|$ matrix *A*.
- Number vertices from 1 to |*V*| in some arbitrary manner.
- *A* is then given by:

$$A[i, j] = a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	1	2	3	4
1	0	1	1	1
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$A = A^T$ for undirected graphs.

Space and Time

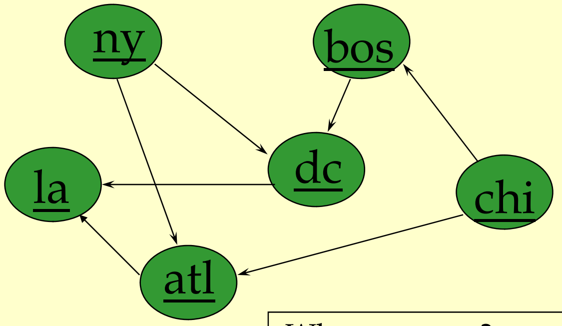
- **Space:** $\Theta(V^2)$.
  - » Not memory efficient for large graphs.
- **Time:** to list all vertices adjacent to u: $\Theta(V)$.
- **Time:** to determine if $(u, v) \in E$: $\Theta(1)$.
- Can store weights instead of bits for weighted graph.

Some graph operations

	adjacency matrix	adjacency lists
insertEdge	O(1)	O(e)
isEdge	O(1)	O(e)
#successors?	O(V)	O(e)
#predecessors?	O(V)	O(E)

---

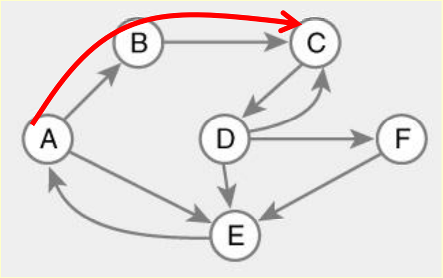
traversing a graph



Where to start?
Will all vertices be visited?
How to prevent multiple visits?

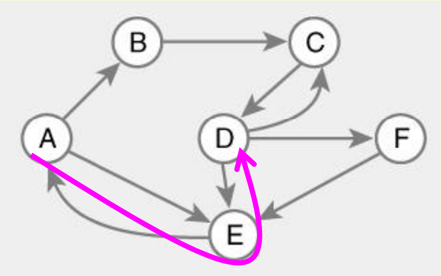
Graph Definitions

- Path
  - » Sequence of nodes $n_1, n_2, \dots n_k$
  - » Edge exists between each pair of nodes n_i, n_{i+1}
  - » Example
    - A, B, C is a path



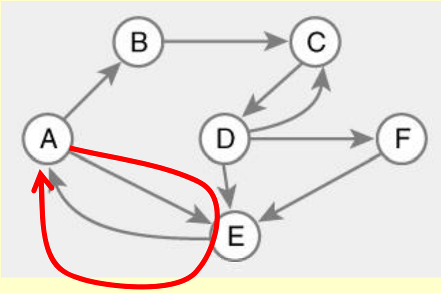
Graph Definitions

- Path
  - Sequence of nodes n_1, n_2, … n_k
  - Edge exists between each pair of nodes n_i , n_{i+1}
  - Example
    - A, B, C is a path
    - A, E, D is not a path



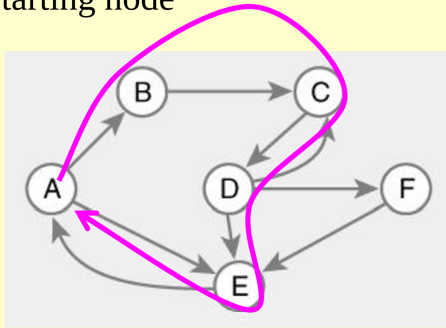
Graph Definitions

- Cycle
  - Path that ends back at starting node
  - Example
    - A, E, A



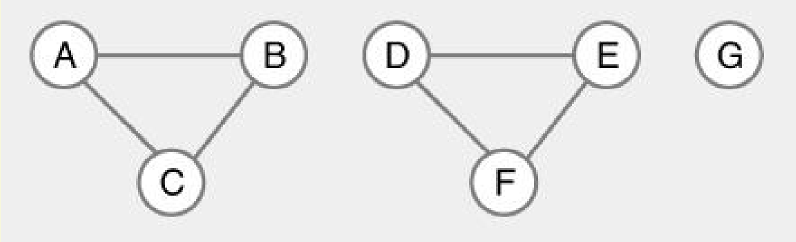
Graph Definitions

- Cycle
  - Path that ends back at starting node
  - Example
    - A, E, A
    - A, B, C, D, E, A
- Simple path
  - No cycles in path
- Acyclic graph
  - No cycles in graph



Graph Definitions

- Reachable
  - Path exists between nodes
- Connected graph
  - Every node is reachable from some node in graph



Unconnected graphs

Graph-searching Algorithms

- Searching a graph:
  - Systematically follow the edges of a graph to visit the vertices of the graph.
- Used to discover the structure of a graph.
- Standard graph-searching algorithms.
  - Breadth-first Search (BFS).
  - Depth-first Search (DFS).

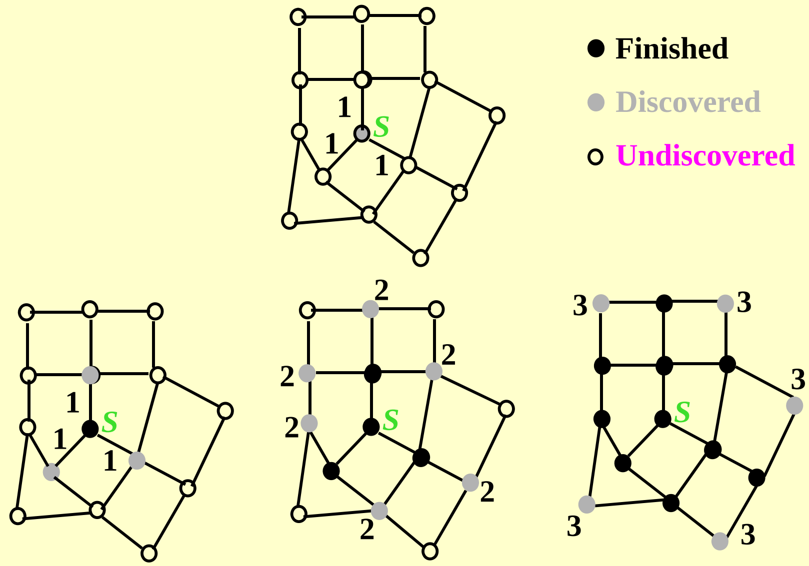
Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - $d[v]$ = distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
  - $\pi[v] = u$ such that (u, v) is last edge on shortest path $s \rightsquigarrow v$.
    - u is v's predecessor.
  - Builds breadth-first tree with root s that contains all reachable vertices.

Breadth-first Search

- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.
  - A vertex is "discovered" the first time it is encountered during the search.
  - A vertex is "finished" if all vertices adjacent to it have been discovered.
- Colors the vertices to keep track of progress.
  - White – Undiscovered.
  - Gray – Discovered but not finished.
  - Black – Finished.

BFS for Shortest Paths



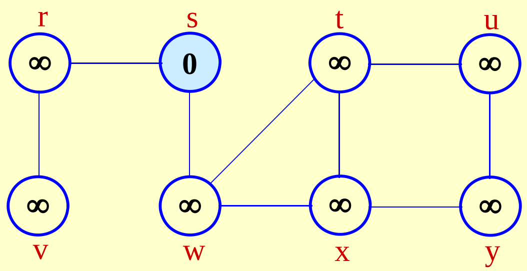
BFS(G,s)

```
for each vertex u in V[G] – {s}
    do color[u] ← white                 initialization
       d[u] ← ∞
       π[u] ← nil
color[s] ← gray
d[s] ← 0                                access source s
π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
    do u ← dequeue(Q)
       for each v in Adj[u]
           do if color[v] = white
                 then color[v] ← gray
                      d[v] ← d[u] + 1
                      π[v] ← u
                      enqueue(Q,v)
       color[u] ← black
```

white: undiscovered
gray: discovered
black: finished

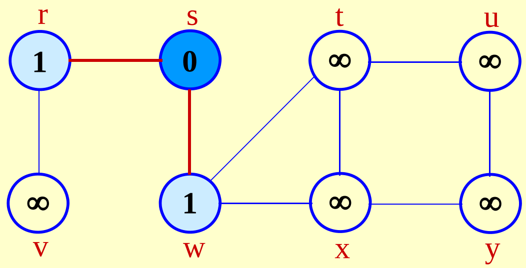
Q: a queue of discovered vertices
color[*v*]: color of v
d[*v*]: distance from s to v
π[*u*]: predecessor of v

Example (BFS)



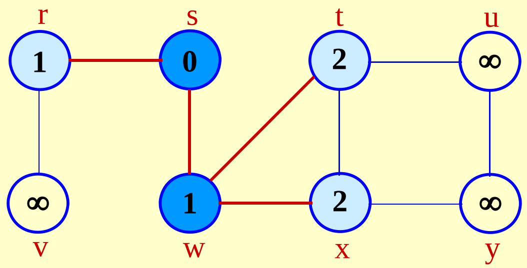
Q: s
0

Example (BFS)



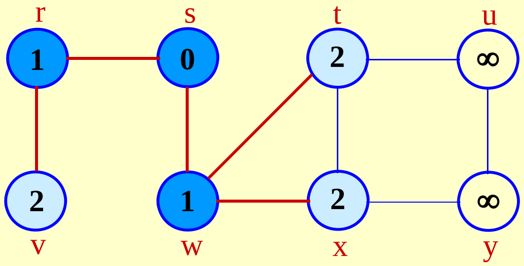
Q: w r
1 1

Example (BFS)



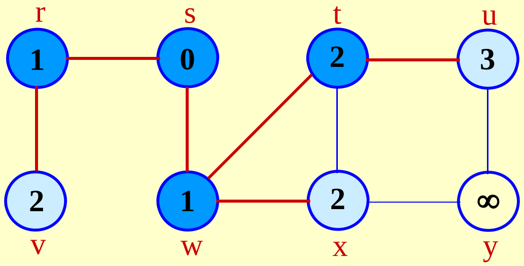
Q: r t x
1 2 2

Example (BFS)



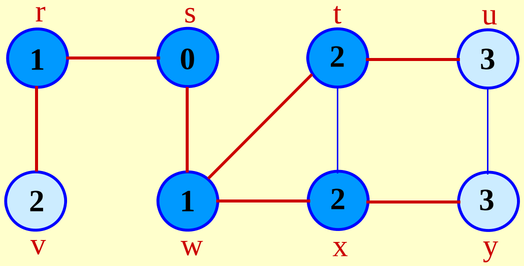
Q: t x v
2 2 2

Example (BFS)



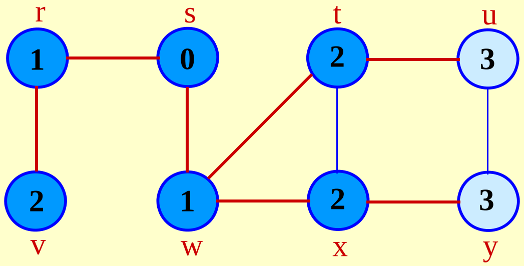
Q: x v u
2 2 3

Example (BFS)



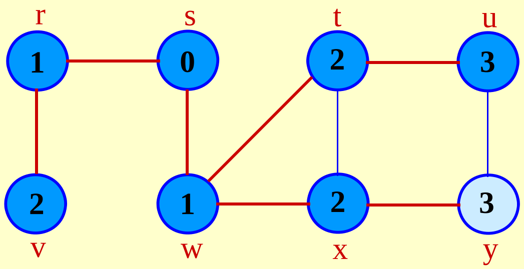
Q: v u y
2 3 3

Example (BFS)



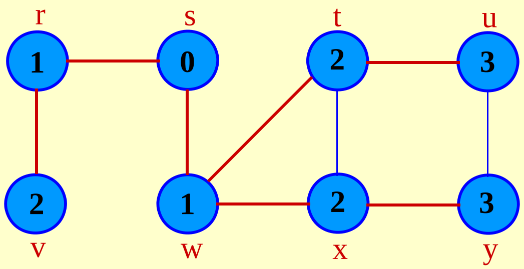
Q: u y
3 3

Example (BFS)



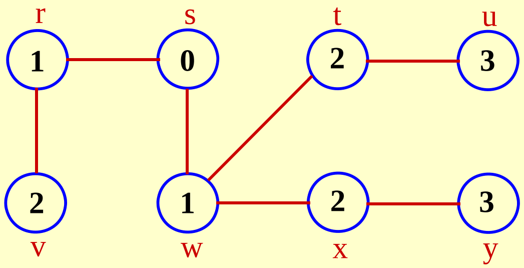
Q: y
3

Example (BFS)



Q: ∅

Example (BFS)



BF Tree

Analysis of BFS

- Initialization takes $O(|V|)$.
- Traversal Loop
  - After initialization, each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(|V|)$.
  - The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $\Theta(|E|)$.
- Summing up over all vertices => total running time of BFS is $O(|V| + |E|)$, linear in the size of the adjacency list representation of graph.

Breadth-first Tree

- For a graph $G = (V, E)$ with source s, the **predecessor subgraph** of G is $G_\pi = (V_\pi, E_\pi)$ where
  - $V_\pi=\{v\in V : \pi[v] \neq nil\} \cup \{s\}$
  - $E_\pi=\{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$
- The predecessor subgraph G_π is a **breadth-first tree** if:
  - V_π consists of the vertices reachable from s and
  - for all $v \in V_\pi$, there is a unique simple path from s to v in G_π that is also a shortest path from s to v in G.
- The edges in E_π are called **tree edges**. $|E_\pi| = |V_\pi| - 1$.

Depth-first Search (DFS)

- Explore edges out of the most recently discovered vertex v.
- When all edges of v have been explored, backtrack to explore other edges leaving the vertex from which v was discovered (its *predecessor*).
- "Search as deep as possible first."
- Continue until all vertices reachable from the original source are discovered.
- If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source.

Depth-first Search

- **Input:** $G = (V, E)$, directed or undirected. No source vertex given!
- **Output:**
  - **2 timestamps** on each vertex. Integers between 1 and 2|V|.
    - $d[v]$ = ***discovery time*** (v turns from white to gray)
    - $f[v]$ = ***finishing time*** (v turns from gray to black)
  - $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u's adjacency list.
- Coloring scheme for vertices as BFS. A vertex is
  - "discovered" the first time it is encountered during the search.
  - A vertex is "finished" if it is a leaf node or all vertices adjacent to it have been finished.

Pseudo-code

DFS(*G*)

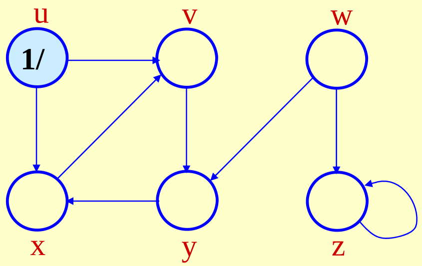
```
for each vertex $u \in V[G]$
    do $color[u] \leftarrow$ white
       $\pi[u] \leftarrow$ NIL
$time \leftarrow 0$
for each vertex $u \in V[G]$
    do if $color[u]$ = white
        then DFS-Visit($u$)
```

Uses a global timestamp ***time***.

DFS-Visit(*u*)

```
$color[u] \leftarrow$ GRAY   // White vertex $u$ has been discovered
$time \leftarrow time + 1$
$d[u] \leftarrow time$
for each $v \in Adj[u]$
    do if $color[v]$ = WHITE
        then $\pi[v] \leftarrow u$
             DFS-Visit($v$)
$color[u] \leftarrow$ BLACK   // Blacken $u$; it is finished.
$f[u] \leftarrow time \leftarrow time + 1$
```

Example (DFS)



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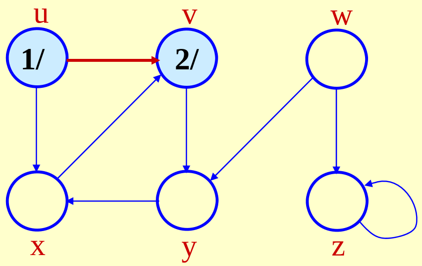

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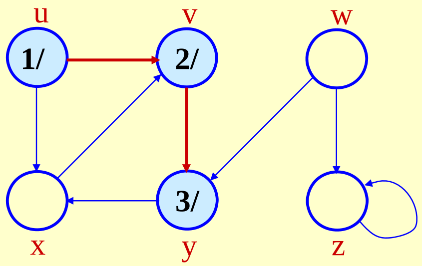

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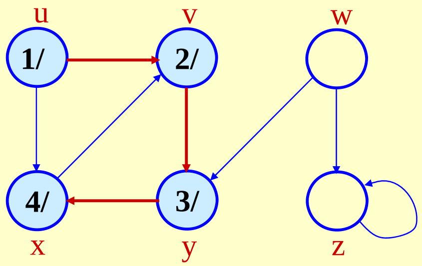

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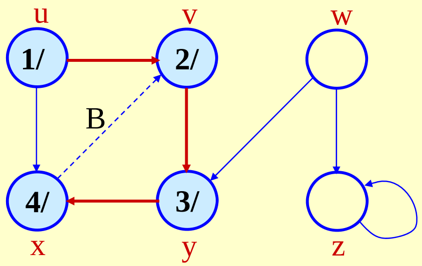

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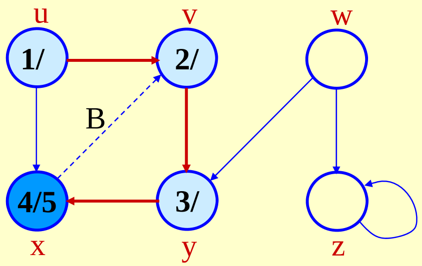

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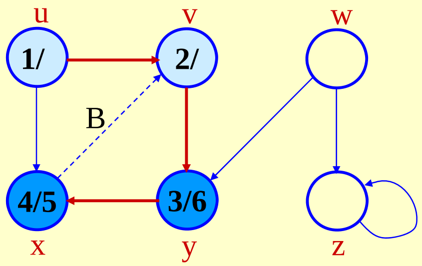

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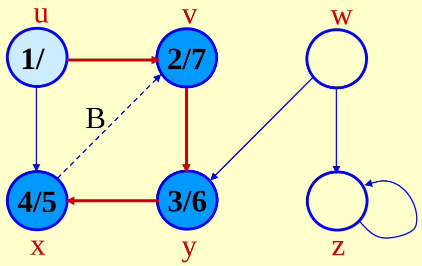

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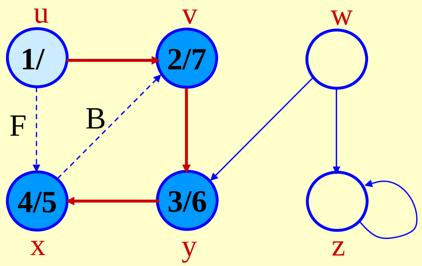

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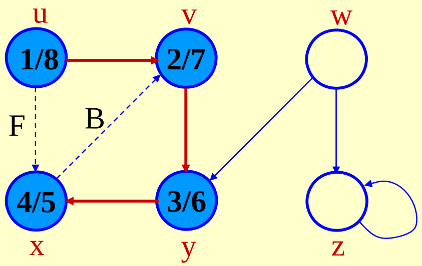

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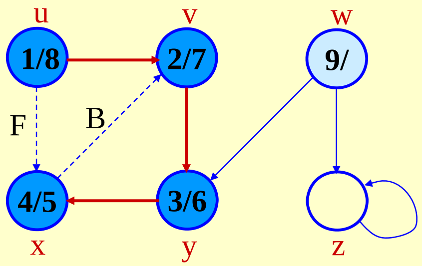

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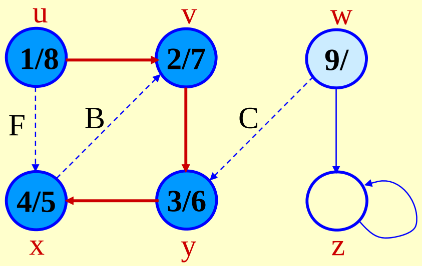

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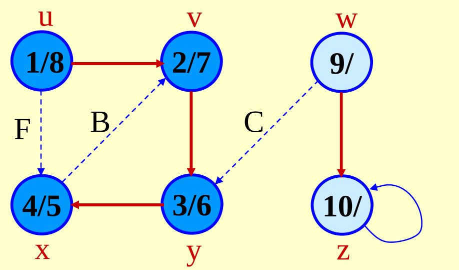

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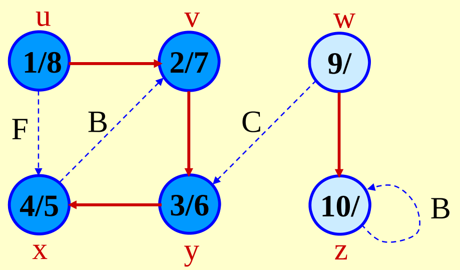

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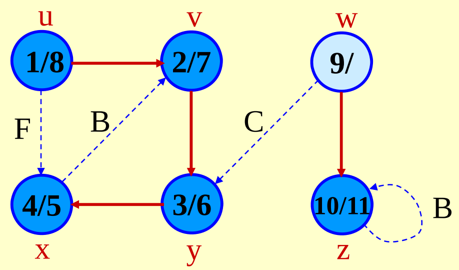

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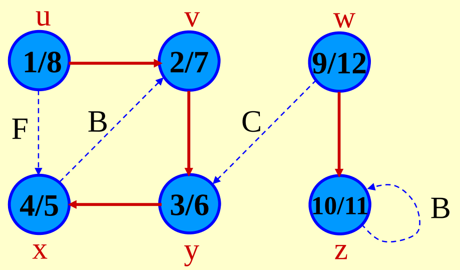


Analysis of DFS

- Loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit.
- DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time. Lines 3-6 of DFS-Visit is executed $|\mathrm{Adj}[v]|$ times. The total cost of executing DFS-Visit is $\sum_{v \in V} |\mathrm{Adj}[v]| = \Theta(E)$
- Total running time of DFS is $\Theta(|V| + |E|)$.

Recursive DFS Algorithm

```
Traverse( )
    for all nodes X
        set X.tag = False
    Visit ( 1st node )
Visit ( X )
    set X.tag = True
    for each successor Y of X
        if (Y.tag = False)
            Visit ( Y )
```

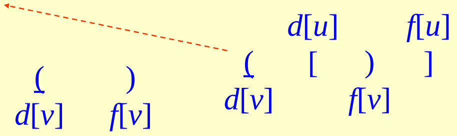
Parenthesis Theorem

Theorem 22.7

For all u, v, exactly one of the following holds:

1. $d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$ and neither u nor v is a descendant of the other.
2. $d[u] < d[v] < f[v] < f[u]$ and v is a descendant of u.
3. $d[v] < d[u] < f[u] < f[v]$ and u is a descendant of v.

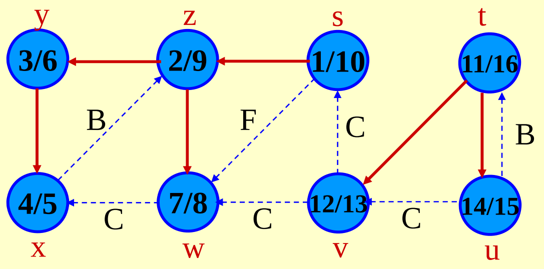
- So $d[u] < d[v] < f[u] < f[v]$ *cannot* happen.
- Like parentheses:
  - OK: ( ) [ ] ([ ]) [( )]
  - Not OK: ([ )] [( ])



Corollary

v is a proper descendant of u if and only if $d[u] < d[v] < f[v] < f[u]$.

Example (Parenthesis Theorem)



(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

Depth-First Trees

- Predecessor subgraph defined slightly different from that of BFS.
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi=\{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq nil\}$.
  - » How does it differ from that of BFS?
  - » The predecessor subgraph G_π forms a *depth-first forest* composed of several *depth-first trees*. The edges in E_π are called *tree edges*.

Definition:
Forest: An acyclic graph G that may be disconnected.

White-path Theorem

Theorem 22.9

v is a descendant of *u* in *DF-tree* if and only if at time $d[u]$, there is a path $u \rightsquigarrow v$ consisting of only white vertices. (Except for *u*, which was *just* colored gray.)

Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring (u, v).
- **Back edge:** (u, v), where *u* is a descendant of *v* (in the depth-first tree).
- **Forward edge:** (u, v), where *v* is a descendant of *u*, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

Theorem:
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

Classifying edges of a digraph

- (u, v) is:
  - » Tree edge – if v is white
  - » Back edge – if v is gray
  - » Forward or cross - if v is black
- (u, v) is:
  - » Forward edge – if v is black and d[u] < d[v] (v was discovered after u)
  - » Cross edge – if v is black and d[u] > d[v] (u discovered after v)

More applications

- **Does directed G contain a directed cycle?** Do DFS if back edges yes. Time O(V+E).
- **Does undirected G contain a cycle?** Same as directed but be careful not to consider (u,v) and (v, u) a cycle.

  Time O(V) since encounter at most |V| edges (if (u, v) and (v, u) are counted as one edge), before cycle is found.
- **Is undirected G a tree?** Do dfsVisit(v). If all vertices are reached and no back edges G is a tree. O(V)

C# Interfaces

```
using System;
using System.Collections.Generic;
using System.Security.Permissions;
[assembly: CLSCompliant(true)]
namespace OhioState.Collections.Graph {
    /// <summary>
    /// IEdge provides a standard interface to specify an edge and any
    /// data associated with an edge within a graph.
    /// </summary>
    /// <typeparam name="N">The type of the nodes in the
        graph.</typeparam>
    /// <typeparam name="E">The type of the data on an edge.</typeparam>
    public interface IEdge<N,E>     {
       /// <summary>
       /// Get the Node label that this edge emanates from.
       /// </summary>
       N From { get; }
       /// <summary>
       /// Get the Node label that this edge terminates at.
       /// </summary>
       N To { get; }
       /// <summary>
       /// Get the edge label for this edge.
       /// </summary>
       E Value { get; }
    }
```

```
/// <summary>
/// The Graph interface
/// </summary>
/// <typeparam name="N">The type associated at each node. Called a
    node or node label</typeparam>
/// <typeparam name="E">The type associated at each edge. Also called
    the edge label.</typeparam>
public interface IGraph<N,E>     {
   /// <summary>
   /// Iterator for the nodes in the graoh.
   /// </summary>
   IEnumerable<N> Nodes { get; }
   /// <summary>
   /// Iterator for the children or neighbors of the specified node.
   /// </summary>
   /// <param name="node">The node.</param>
   /// <returns>An enumerator of nodes.</returns>
   IEnumerable<N> Neighbors(N node);
  /// <summary>
   /// Iterator over the parents or immediate ancestors of a node.
   /// </summary>
   /// <remarks>May not be supported by all graphs.</remarks>
   /// <param name="node">The node.</param>
   /// <returns>An enumerator of nodes.</returns>
   IEnumerable<N> Parents(N node);
```

C# Interfaces

```
/// <summary>
/// Iterator over the emanating edges from a node.
/// </summary>
/// <param name="node">The node.</param>
/// <returns>An enumerator of nodes.</returns>
IEnumerable<IEdge<N, E>> OutEdges(N node);
/// <summary>
/// Iterator over the in-coming edges of a node.
/// </summary>
/// <remarks>May not be supported by all graphs.</remarks>
/// <param name="node">The node.</param>
/// <returns>An enumerator of edges.</returns>
IEnumerable<IEdge<N, E>> InEdges(N node);
/// <summary>
/// Iterator for the edges in the graph, yielding IEdge's
/// </summary>
IEnumerable<IEdge<N, E>> Edges { get; }
/// <summary>
/// Tests whether an edge exists between two nodes.
/// </summary>
/// <param name="fromNode">The node that the edge emanates
 from.</param>
/// <param name="toNode">The node that the edge terminates
 at.</param>
/// <returns>True if the edge exists in the graph. False
 otherwise.</returns>
bool ContainsEdge(N fromNode, N toNode);
/// <summary>
/// Gets the label on an edge.
/// </summary>
/// <param name="fromNode">The node that the edge emanates
 from.</param>
/// <param name="toNode">The node that the edge terminates
 at.</param>
```

```
        /// <returns>The edge.</returns>
        E GetEdgeLabel(N fromNode, N toNode);
        /// <summary>
        /// Exception safe routine to get the label on an edge.
        /// </summary>
        /// <param name="fromNode">The node that the edge emanates
         from.</param>
        /// <param name="toNode">The node that the edge terminates
         at.</param>
        /// <param name="edge">The resulting edge if the method was
         successful. A default
        /// value for the type if the edge could not be found.</param>
        /// <returns>True if the edge was found. False otherwise.</returns>
        bool TryGetEdge(N fromNode, N toNode, out E edge);
    }
}
```

C# Interfaces

```
using System;
  namespace OhioState.Collections.Graph {
    /// <summary>
    /// Graph interface for graphs with finite size.
    /// </summary>
    /// <typeparam name="N">The type associated at each node. Called a node or node
      label</typeparam>
    /// <typeparam name="E">The type associated at each edge. Also called the edge
      label.</typeparam>
    /// <seealso cref="IGraph{N, E}"/>
    public interface IFiniteGraph<N, E> : IGraph<N, E>     {
        /// <summary>
        /// Get the number of edges in the graph.
        /// </summary>
        int NumberOfEdges { get; }
        /// <summary>
        /// Get the number of nodes in the graph.
        /// </summary>
        int NumberOfNodes { get; }
    }
 }
```