Introduction to Algorithms

Hash Tables

CSE 680
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Motivation

- Arrays provide an indirect way to access a **set**.
- Many times we need an association between two sets, or a set of **keys** and associated data.
- Ideally we would like to access this data directly with the keys.
- We would like a data structure that supports fast search, insertion, and deletion.
  - Do not usually care about sorting.
- The abstract data type is usually called a **Dictionary** or **Partial Map**
  - `float googleStockPrice = stocks["Goog"].CurrentPrice;`

Dictionaries

- What is the best way to implement this?
  - Linked Lists?
  - Double Linked Lists?
  - Queues?
  - Stacks?
  - Multiple indexed arrays (e.g., data[key[i]])?
- To answer this, ask what the complexity of the operations are:
  - Insertion
  - Deletion
  - Search

Direct Addressing

- Let's look at an easy case, suppose:
  - The range of keys is 0..m-1
  - Keys are distinct
- Possible solution
  - Set up an array T[0..m-1] in which
    - T[i] = x if $x \in T$ and key[x] = i
    - T[i] = NULL otherwise
  - This is called a *direct-address table*
    - Operations take O(1) time!
    - *So what's the problem?*

Direct Addressing

- Direct addressing works well when the range *m* of keys is relatively small
- But what if the keys are 32-bit integers?
  - Problem 1: direct-address table will have 2^{32} entries, more than 4 billion
  - Problem 2: even if memory is not an issue, the time to initialize the elements to NULL may be
- Solution: map keys to smaller range 0..*p*-1
  - Desire $p = O(m)$.

Hash Table

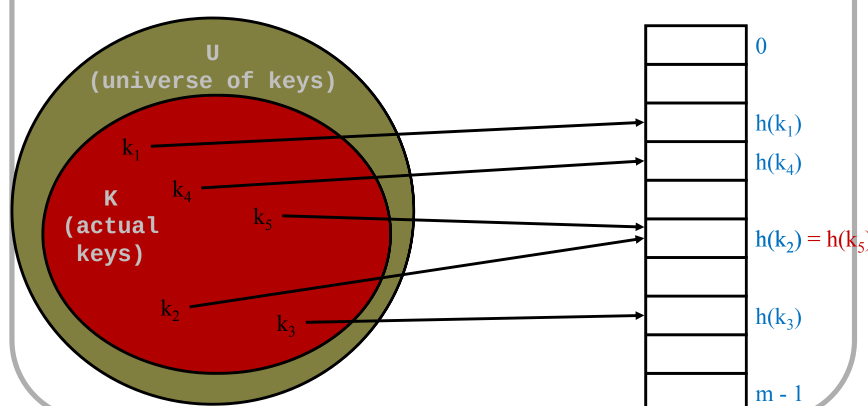
- Hash Tables provide ***O***(1) support for all of these operations!
- The key is rather than index an array directly, index it through some function, ***h***(*x*), called a *hash function*.
  - myArray[*h*(index)]
- Key questions:
  - What is the set that the *x* comes from?
  - What is *h()* and what is its range?

Hash Table

- Consider this problem:
  - If I know a prior the *m* keys from some finite set **U**, is it possible to develop a function *h(x)* that will uniquely map the m keys onto the set of numbers 0..*m*-1?

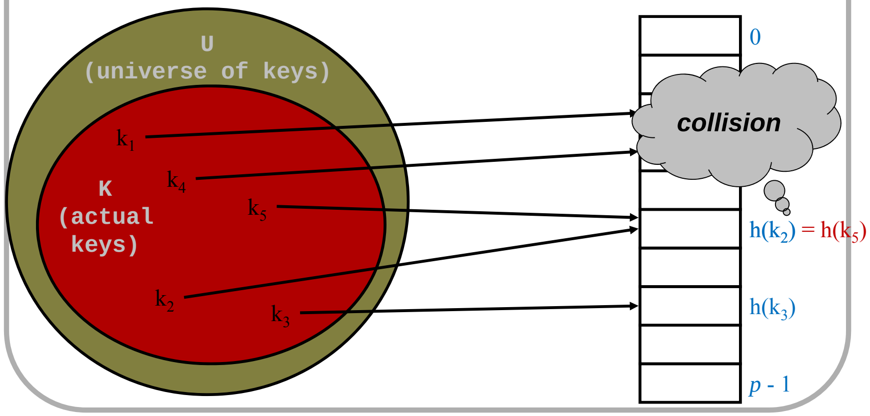
Hash Functions

- In general a difficult problem. Try something simpler.



Hash Functions

- A **collision** occurs when *h(x)* maps two keys to the same location.

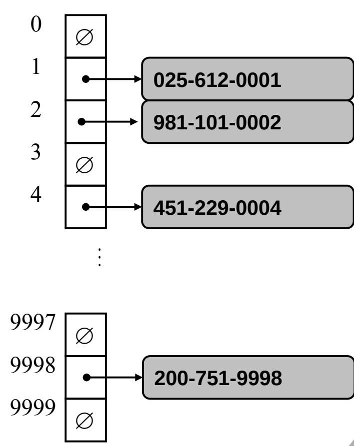


Hash Functions

- A hash function, $\boldsymbol{h}$, maps keys of a given type to integers in a fixed interval $[0, N-1]$
- Example:
  $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{x} \bmod \boldsymbol{N}$
  is a hash function for integer keys
- The integer $\boldsymbol{h}(\boldsymbol{x})$ is called the hash value of $\boldsymbol{x}$.
- A hash table for a given key type consists of
  - Hash function $\boldsymbol{h}$
  - Array (called table) of size $\boldsymbol{N}$
- The goal is to store item $(\boldsymbol{k}, \boldsymbol{o})$ at index $\boldsymbol{i} = \boldsymbol{h}(\boldsymbol{k})$

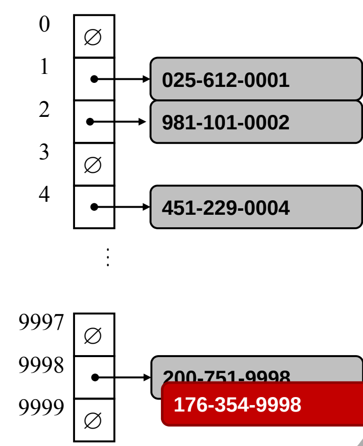
Example

- We design a hash table storing employees records using their social security number, SSN as the key.
  - SSN is a nine-digit positive integer
- Our hash table uses an array of size $N = 10{,}000$ and the hash function $\boldsymbol{h}(\boldsymbol{x}) =$ last four digits of $\boldsymbol{x}$



Example

- Our hash table uses an *array* of size **N = 100.**
- We have ***n*** **= 49** employees.
  - Need a method to handle ***collisions***.
- As long as the chance for collision is low, we can achieve this goal.
- Setting **N = 1000** and looking at the last four digits will *reduce* the chance of collision.

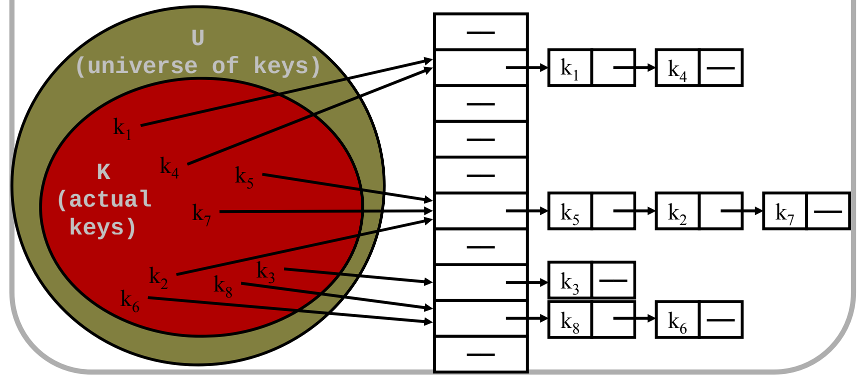


Collisions

- Can collisions be avoided?
  - In general, no. See *perfect hashing* for the case were the set of keys is static (not covered).
- Two primary techniques for resolving collisions:
  - **Chaining** – keep a collection at each key slot.
  - **Open addressing** – if the current slot is full use the ***next open*** one.

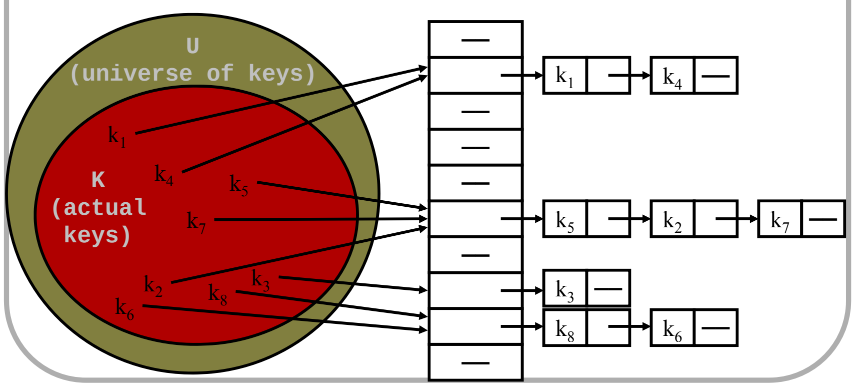
Chaining

- Chaining puts elements that hash to the same slot in a linked list:



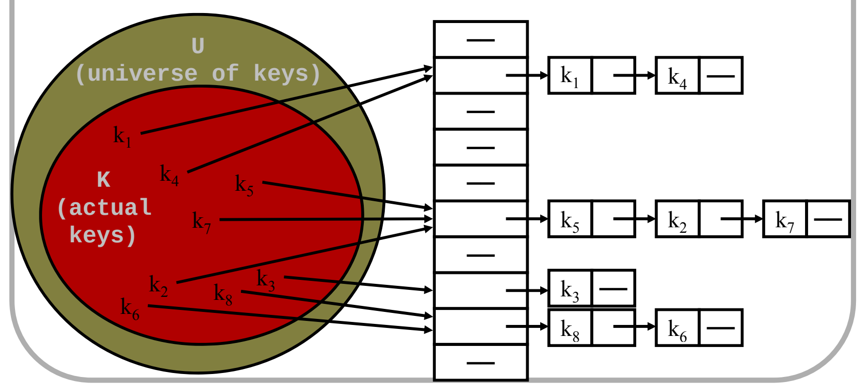
Chaining

- *How do we insert an element?*



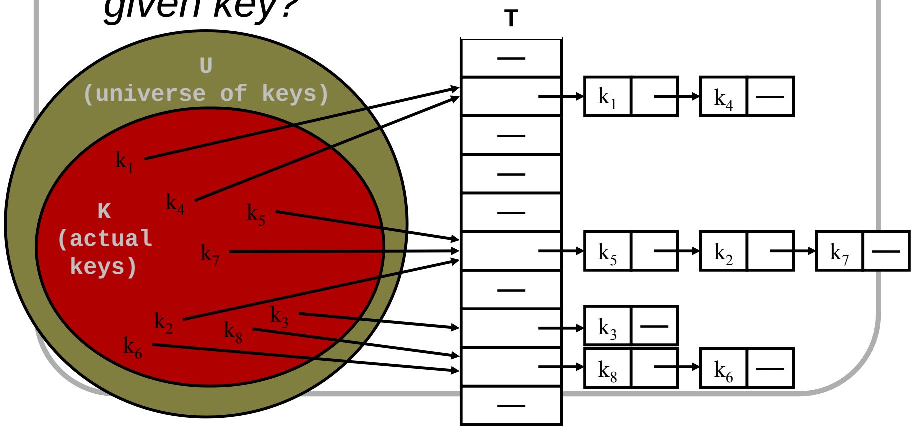
Chaining

- *How do we delete an element?*
  - *Do we need a doubly-linked list for efficient delete?*



Chaining

- *How do we search for a element with a given key?*



Open Addressing

- Basic idea:
  - To insert: if slot is full, try another slot, …, until an open slot is found (***probing***)
  - To search, follow same sequence of probes as would be used when inserting the element
    - If reach element with correct key, return it
    - If reach a NULL pointer, element is not in table
- Good for fixed sets (adding but no deletion)
  - Example: spell checking

Open Addressing

- The colliding item is placed in a different cell of the table.
  - No dynamic memory.
  - Fixed Table size.
- **Load factor:** *n/N*, where *n* is the number of items to store and N the size of the hash table.
  - Cleary, $n \le N$, or $n/N \le 1$.
- To get a reasonable performance, $n/N < 0.5$.

Probing

- They key question is what should the next cell to try be?
- Random would be great, but we need to be able to repeat it.
- Three common techniques:
  - Linear Probing (useful for discussion only)
  - Quadratic Probing
  - Double Hashing

Linear Probing

- **Linear probing** handles collisions by placing the colliding item in the ***next*** (circularly) available table cell.
- Each table cell inspected is referred to as a ***probe.***
- Colliding items lump together, causing future collisions to cause a longer sequence of probes.

- Example:
  - $h(x) = x \bmod 13$
  - Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

0	1	2	3	4	5	6	7	8	9	10	11	12

⇩

		41			18	44	59	32	22	31	73	
0	1	2	3	4	5	6	7	8	9	10	11	12

Search with Linear Probing

- Consider a hash table A that uses linear probing
- get(k)
  - We start at cell $h(k)$
  - We probe consecutive locations until one of the following occurs
    - An item with key k is found, or
    - *An empty cell is found*, or
    - N cells have been unsuccessfully probed
  - To ensure the efficiency, if k is not in the table, we want to find an empty cell as soon as possible. The load factor can NOT be close to 1.

```
Algorithm
  i ← h(k)
  p ← 0
  repeat
     c ← A[i]
     if c = ∅
         return null
     else if c.key () = k
         return c.element()
     else
         i ← (i + 1) mod N
         p ← p + 1
  until  p = N
  return null
```

Linear Probing

- Search for key=20.
  - h(20)=20 mod 13 =7.
  - Go through rank 8, 9, …, 12, 0.
- Search for key=15
  - h(15)=15 mod 13=2.
  - Go through rank 2, 3 and return null.

- Example:
  - $h(x) = x \bmod 13$
  - Insert keys 18, 41, 22, 44, 59, 32, 31, 73, 12, 20 in this order

0	1	2	3	4	5	6	7	8	9	10	11	12

⇩

20		41			18	44	59	32	22	31	73	12
0	1	2	3	4	5	6	7	8	9	10	11	12

Updates with Linear Probing

- To handle insertions and deletions, we introduce a special object, called ***AVAILABLE***, which replaces deleted elements
- remove(k)
  - We search for an entry with key k
  - If such an entry (k, o) is found, we replace it with the special item ***AVAILABLE*** and we return element o
  - ***Have to modify other methods to skip available cells.***
- put(k, o)
  - We throw an exception if the table is full
  - We start at cell $h(k)$
  - We probe consecutive cells until one of the following occurs
    - A cell i is found that is either empty or stores ***AVAILABLE***, or
    - N cells have been unsuccessfully probed
  - We store entry (k, o) in cell i

Quadratic Probing

- Primary clustering occurs with linear probing because the same linear pattern:
  - if a bin is inside a cluster, then the next bin must either:
    - also be in that cluster, or
    - expand the cluster
- Instead of searching forward in a linear fashion, consider searching forward using a quadratic function

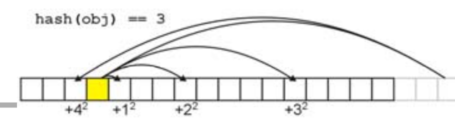
Quadratic Probing

- Suppose that an element should appear in bin h:
  - if bin h is occupied, then check the following sequence of bins:

$$h + 1^2,\ \ h + 2^2,\ \ h + 3^2,\ \ h + 4^2,\ \ h + 5^2,\ \ldots$$

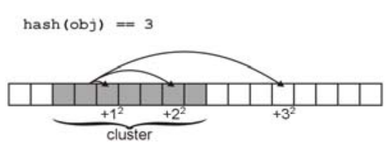
$$h + 1,\ \ h + 4,\ \ h + 9,\ \ h + 16,\ \ h + 25,\ \ldots$$

- For example, with $M = 17$:



Quadratic Probing

- If one of $h + i^2$ falls into a cluster, this does not imply the next one will



Quadratic Probing

- For example, suppose an element was to be inserted in bin 23 in a hash table with 31 bins
- The sequence in which the bins would be checked is:

23, 24, 27, 1, 8, 17, 28, 10, 25, 11, 30, 20, 12, 6, 2, 0

Quadratic Probing

- Even if two bins are initially close, the sequence in which subsequent bins are checked varies greatly
- Again, with $M = 31$ bins, compare the first 16 bins which are checked starting with 22 and 23:

22, 23, 26, 0, 7, 16, 27, 9, 24, 10, 29, 19, 11, 5, 1, 30
23, 24, 27, 1, 8, 17, 28, 10, 25, 11, 30, 20, 12, 6, 2, 0

Quadratic Probing

- Thus, quadratic probing solves the problem of primary clustering
- Unfortunately, there is a second problem which must be dealt with
- Suppose we have $M = 8$ bins:

$$1^2 \equiv 1,\ 2^2 \equiv 4,\ 3^2 \equiv 1$$

- In this case, we are checking bin $h + 1$ twice having checked only one other bin

Quadratic Probing

- Unfortunately, there is no guarantee that

$$h + i^2 \bmod M$$

  will cycle through $0, 1, ..., M - 1$
- Solution:
  - require that M be prime
  - in this case, $h + i^2 \bmod M$ for $i = 0, ..., (M - 1)/2$ will cycle through exactly $(M + 1)/2$ values before repeating

Quadratic Probing

- Example with $M = 11$:

  $0, 1, 4, 9, 16 \equiv 5, 25 \equiv 3, 36 \equiv 3$
- With $M = 13$:

  $0, 1, 4, 9, 16 \equiv 3, 25 \equiv 12, 36 \equiv 10, 49 \equiv 10$
- With $M = 17$:

  $0, 1, 4, 9, 16, 25 \equiv 8, 36 \equiv 2, 49 \equiv 15, 64 \equiv 13, 81 \equiv 13$

Quadratic Probing

- Thus, quadratic probing avoids primary clustering
- Unfortunately, we are not guaranteed that we will use all the bins
- In reality, if the hash function is reasonable, this is not a significant problem until λ approaches 1

Secondary Clustering

- The phenomenon of primary clustering will not occur with quadratic probing
- However, if multiple items all hash to the same initial bin, the same sequence of numbers will be followed
- This is termed *secondary clustering*
- The effect is less significant than that of primary clustering

Double Hashing

- Use two hash functions
- If **M** is prime, eventually will examine every position in the table
- double_hash_insert(K)

```
if(table is full) error
probe = h1(K)
offset = h2(K)
while (table[probe] occupied)
    probe = (probe + offset) mod M
table[probe] = K
```

Double Hashing

- Many of same (dis)advantages as linear probing
- Distributes keys more uniformly than linear probing does
- Notes:
  - h2(x) should never return zero.
  - M should be prime.

Double Hashing Example

- h1(K) = K mod 13
- h2(K) = 8 - K mod 8
  - we want h2 to be an offset to add
  - 18 41 22 44 59 32 31 73

0	1	2	3	4	5	6	7	8	9	10	11	12

44		41	73		18	32	53	31	22			
0	1	2	3	4	5	6	7	8	9	10	11	12

Open Addressing Summary

- In general, the hash function contains two arguments now:
  - Key value
  - Probe number

  h(k,p), p=0,1,...,m-1
- Probe sequences

  <h(k,0), h(k,1), ..., h(k,m-1)>
  - Should be a permutation of <0,1,...,m-1>
  - There are m! possible permutations
  - Good hash functions should be able to produce all m! probe sequences

Open Addressing Summary

- None of the methods discussed can generate more than ***m^2*** different probing sequences.
- Linear Probing:
  - Clearly, only *m* probe sequences.
- Quadratic Probing:
  - The initial key determines a fixed probe sequence, so only m distinct probe sequences.
- Double Hashing
  - Each possible pair $(h_1(k),h_2(k))$ yields a distinct probe, so m^2 permutations.

Choosing A Hash Function

- Clearly choosing the hash function well is crucial.
  - *What will a worst-case hash function do?*
  - *What will be the time to search in this case?*
- *What are desirable features of the hash function?*
  - Should distribute keys uniformly into slots
  - Should not depend on patterns in the data

From Keys to Indices

- A hash function is usually the composition of two maps:
  - **hash code map**: key → integer
  - **compression map**: integer → [0, N – 1]
- An essential requirement of the hash function is to ***map equal keys to equal indices***
- A "good" hash function minimizes the probability of collisions

Java Hash

- Java provides a hashCode() method for the Object class, which typically returns the 32-bit memory address of the object.
- This default hash code would work poorly for Integer and String objects
- The hashCode() method should be suitably redefined by classes.

Popular Hash-Code Maps

- ***Integer cast***: for numeric types with 32 bits or less, we can reinterpret the bits of the number as an int
- ***Component sum***: for numeric types with more than 32 bits (e.g., long and double), we can add the 32-bit components.

Popular Hash-Code Maps

- ***Polynomial accumulation***: for strings of a natural language, combine the character values (ASCII or Unicode) a_0 a_1 ... a_{n-1} by viewing them as the coefficients of a polynomial: $a_0 + a_1 x + ... + x_{n-1} a_{n-1}$

Popular Hash-Code Maps

- The polynomial is computed with ***Horner's rule***, ignoring overflows, at a fixed value x:
  $a_0 + x\,(a_1 + x\,(a_2 + \ldots\ x\,(a_{n-2} + x\,a_{n-1}\ )\ \ldots\ ))$
- The choice x = 33, 37, 39, or 41 gives at most 6 collisions on a vocabulary of 50,000 English words

- Why is the component-sum hash code bad for strings?

Random Hashing

- Random hashing
  - Uses a simple random number generation technique
  - Scatters the items "randomly" throughout the hash table

Popular Compression Maps

- ***Division***: h(k) = |k| **mod** *N*
  - the choice *N* =2 *k* is bad because not all the bits are taken into account
  - the table size *N* is usually chosen as a prime number
  - certain patterns in the hash codes are propagated
- ***Multiply, Add, and Divide*** (MAD):
  - h(k) = |*ak* + b| **mod** *N*
  - eliminates patterns provided *a* **mod** $N \neq 0$
  - same formula used in linear congruential (pseudo) random number generators

The Division Method

- $h(k) = k \bmod m$
  - In words: hash *k* into a table with *m* slots using the slot given by the remainder of *k* divided by *m*
- *What happens to elements with adjacent values of k?*
- *What happens if m is a power of 2 (say 2^P)?*
- *What if m is a power of 10?*
- Upshot: pick table size *m* = prime number not too close to a power of 2 (or 10)

The Multiplication Method

- For a constant A, $0 < A < 1$:
- $h(k) = \lfloor m\, \underbrace{(kA - \lfloor kA \rfloor)}_{\text{What does this term represent?}} \rfloor$

The Multiplication Method

- For a constant A, $0 < A < 1$:
- $h(k) = \lfloor m\, \underbrace{(kA - \lfloor kA \rfloor)}_{\text{Fractional part of kA}} \rfloor$
- Choose $m = 2^P$
- Choose A not too close to 0 or 1
- Knuth: Good choice for $A = (\sqrt{5} - 1)/2$

Analysis of Chaining

- Assume *simple uniform hashing*: each key in table is equally likely to be hashed to any slot.
- Given n keys and m slots in the table: the *load factor* $\alpha = n/m$ = average # keys per slot.

Analysis of Chaining

- *What will be the **average** cost of an unsuccessful search for a key?*
- $O(1+\alpha)$

Analysis of Chaining

- *What will be the **average** cost of an unsuccessful search for a key?*
- O(1+α)

Analysis of Chaining

- *What will be the **average** cost of a successful search?*
- O(1 + α/2) = O(1 + α)

Analysis of Chaining

- So the cost of searching = O(1 + α)
- *If the number of keys n is proportional to the number of slots in the table, what is α?*
- A: α = O(1)
  - In other words, we can make the expected cost of searching constant if we make α constant

Analysis of Open Addressing

- Consider the load factor, α, and assume each key is uniformly hashed.
- Probability that we hit an occupied cell is then α.
- Probability that we the next probe hits an occupied cell is also α.
- Will terminate if an unoccupied cell is hit: $\alpha(1- \alpha)$.
- From Theorem 11.6, the expected number of probes in an ***unsuccessful*** search is at most 1/(1- α).
- Theorem 11.8: Expected number of probes in a successful search is at most:

$$\frac{1}{\alpha}\ln\left(\frac{1}{1-\alpha}\right)$$