Introduction to Algorithms
Quicksort

CSE 680
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Sorting Review

- Insertion Sort
  - $T(n) = \Theta(n^2)$
  - In-place
- Merge Sort
  - $T(n) = \Theta(n \lg(n))$
  - Not in-place
- Selection Sort (from homework)
  - $T(n) = \Theta(n^2)$
  - In-place
- Heap Sort
  - $T(n) = \Theta(n \lg(n))$
  - In-place

Seems pretty good.
Can we do better?

Sorting

- Assumptions
  1. No knowledge of the keys or numbers we are sorting on.
  2. Each key supports a comparison interface or operator.
  3. Sorting entire records, as opposed to numbers, is an implementation detail.
  4. Each key is unique (just for convenience).

Comparison Sorting

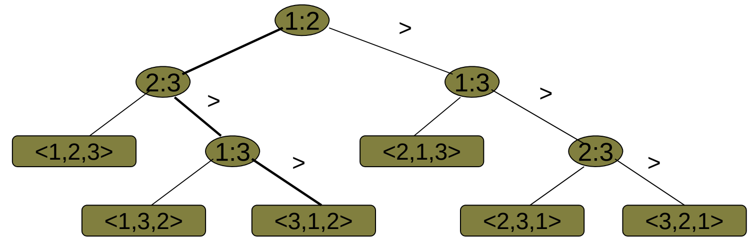
Comparison Sorting

- Given a set of *n* values, there can be *n!* permutations of these values.
- So if we look at the behavior of the sorting algorithm over all possible *n!* inputs we can determine the worst-case complexity of the algorithm.

Decision Tree

- Decision tree model
  - Full binary tree
    - A **full binary tree** (sometimes **proper binary tree** or **2-tree**) is a tree in which every node other than the leaves has two children
  - Internal node represents a comparison.
    - Ignore control, movement, and all other operations, just see comparison
  - Each leaf represents one possible result (a permutation of the elements in sorted order).
  - The height of the tree (i.e., longest path) is the lower bound.

Decision Tree Model



Internal node $i{:}j$ indicates comparison between a_i and a_j.
suppose three elements < $a1$, $a2$, $a3$> with instance <6,8,5>
Leaf node <$\pi(1)$, $\pi(2)$, $\pi(3)$> indicates ordering $a_{\pi(1)}$ $a_{\pi(2)}$ $a_{\pi(3)}$.
Path of **bold lines** indicates sorting path for <6,8,5>.
There are total 3!=6 possible permutations (paths).

Decision Tree Model

- The longest path is the worst case number of comparisons. The length of the longest path is the height of the decision tree.
- **Theorem 8.1**: Any comparison sort algorithm requires $\Omega(n\lg n)$ comparisons in the worst case.
- **Proof:**
  - Suppose height of a decision tree is h, and number of paths (i,e,, permutations) is $n!$.
  - Since a binary tree of height h has at most 2^h leaves,
    - $n! \le 2^h$, so $h \ge \lg(n!) \ge \Omega(n\lg n)$ (By equation 3.18).
- That is to say: **any comparison sort in the worst case needs at least *n*lg *n* comparisons.**

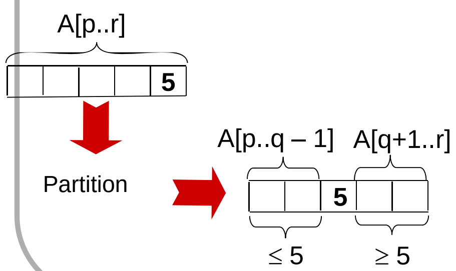
QuickSort Design

- Follows the **divide-and-conquer** paradigm.
- ***Divide*: Partition** (separate) the array $A[p..r]$ into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$.
  - Each element in $A[p..q-1] < A[q]$.
  - $A[q] <$ each element in $A[q+1..r]$.
  - Index q is computed as part of the partitioning procedure.
- ***Conquer*:** Sort the two subarrays by recursive calls to quicksort.
- ***Combine*:** The subarrays are sorted in place – no work is needed to combine them.
- How do the divide and combine steps of quicksort compare with those of merge sort?

Pseudocode

Quicksort(A, p, r)
```
Quicksort(A, p, r)
    if p < r then
        q := Partition(A, p, r);
        Quicksort(A, p, q – 1);
        Quicksort(A, q + 1, r)
```

```
Partition(A, p, r)
    x, i := A[r], p – 1;
    for j := p to r – 1 do
        if A[j] ≤ x then
            i := i + 1;
            A[i] ↔ A[j]
    A[i + 1] ↔ A[r];
    return i + 1
```



Example

```
                        p                          r
initially:              2  5  8  3  9  4  1  7  10  6      note: pivot (x) = 6
                      i j

next iteration:         2  5  8  3  9  4  1  7  10  6
                        i  j

next iteration:         2  5  8  3  9  4  1  7  10  6
                           i  j

next iteration:         2  5  8  3  9  4  1  7  10  6
                           i     j

next iteration:         2  5  3  8  9  4  1  7  10  6
                              i     j
```

```
Partition(A, p, r)
    x, i := A[r], p – 1;
    for j := p to r – 1 do
        if A[j] ≤ x then
            i := i + 1;
            A[i] ↔ A[j]
    A[i + 1] ↔ A[r];
    return i + 1
```

Example (Continued)

```
next iteration:     2  5  3  8  9  4  1  7  10  6
                          i     j
next iteration:     2  5  3  8  9  4  1  7  10  6
                          i        j
next iteration:     2  5  3  4  9  8  1  7  10  6
                             i        j
next iteration:     2  5  3  4  1  8  9  7  10  6
                                i        j
next iteration:     2  5  3  4  1  8  9  7  10  6
                                i           j
next iteration:     2  5  3  4  1  8  9  7  10  6
                                i               j
after final swap:   2  5  3  4  1  6  9  7  10  8
                                i               j
```

```
Partition(A, p, r)
    x, i := A[r], p – 1;
    for j := p to r – 1 do
        if A[j] ≤ x then
            i := i + 1;
            A[i] ↔ A[j]
    A[i + 1] ↔ A[r];
    return i + 1
```

Partitioning

- Select the last element A[*r*] in the subarray *A*[*p*..*r*] as the *pivot* – the element around which to partition.
- As the procedure executes, the array is partitioned into four (possibly empty) regions.
  1. *A*[*p*..*i*] — All entries in this region are **< *pivot***.
  2. *A*[*i*+1..*j* – 1] — All entries in this region are **> *pivot***.
  3. *A*[*r*] = *pivot*.
  4. *A*[*j*..*r* – 1] — Not known how they compare to *pivot*.
- The above hold before each iteration of the *for* loop, and constitute a *loop invariant.* (4 is not part of the loopi.)

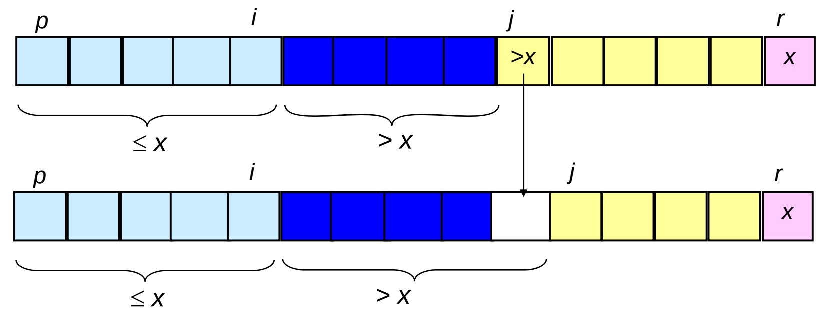
Correctness of Partition

- Use loop invariant.
- **Initialization:**
  - Before first iteration
    - *A*[*p*..*i*] and *A*[*i*+1..*j* – 1] are empty – Conds. 1 and 2 are satisfied (trivially).
    - *r* is the index of the *pivot*
      - Cond. 3 is satisfied.
- **Maintenance:**
  - **Case 1:** *A*[*j*] > *x*
    - Increment *j* only.
    - Loop Invariant is maintained.

```
Partition(A, p, r)
    x, i := A[r], p – 1;
    for j := p to r – 1 do
        if A[j] ≤ x then
            i := i + 1;
            A[i] ↔ A[j]
    A[i + 1] ↔ A[r];
    return i + 1
```

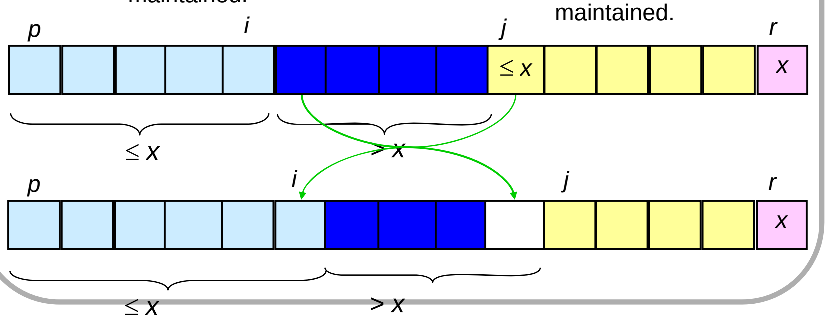
Correctness of Partition

Case 1:



Correctness of Partition

- **Case 2:** *A*[*j*] ≤ *x*
  - Increment *i*
  - Swap *A*[*i*] and *A*[*j*]
    - Condition 1 is maintained.
- Increment *j*
  - Condition 2 is maintained.
- *A*[*r*] is unaltered.
  - Condition 3 is maintained.



Correctness of Partition

- **Termination:**
  - When the loop terminates, *j* = *r*, so all elements in *A* are partitioned into one of the three cases:
    - *A*[*p*..*i*] ≤ ***pivot***
    - *A*[*i*+1..*j* – 1] > ***pivot***
    - *A*[*r*] = ***pivot***
- The last two lines swap *A*[*i*+1] and *A*[*r*].
  - *Pivot* moves from the end of the array to between the two subarrays.
  - Thus, procedure *partition* correctly performs the divide step.

Complexity of Partition

- PartitionTime(n) is given by the number of iterations in the *for* loop.
- $\Theta(n)$: $n = r - p + 1$.

```
Partition(A, p, r)
    x, i := A[r], p – 1;
    for j := p to r – 1 do
        if A[j] ≤ x then
            i := i + 1;
            A[i] ↔ A[j]
    A[i + 1] ↔ A[r];
    return i + 1
```

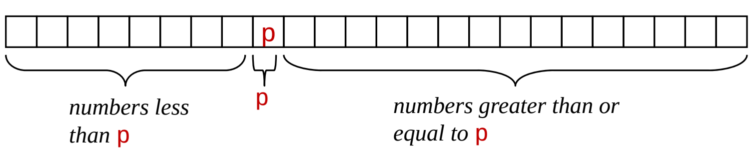
Quicksort Overview

- To sort a[left...right]:

```
if left < right:
    Partition a[left...right] such that:
      all a[left...p-1] are less than a[p], and
      all a[p+1...right] are >= a[p]
    Quicksort a[left...p-1]
    Quicksort a[p+1...right]
Terminate
```

Partitioning in Quicksort

- A key step in the Quicksort algorithm is partitioning the array
  - We choose some (any) number p in the array to use as a pivot
  - We partition the array into three parts:



Alternative Partitioning

- Choose an array value (say, the first) to use as the pivot
- Starting from the left end, find the first element that is greater than or equal to the pivot
- Searching backward from the right end, find the first element that is less than the pivot
- Interchange (swap) these two elements
- Repeat, searching from where we left off, until done

Alternative Partitioning

- To partition a[left...right]:

  Set pivot = a[left], l = left + 1, r = right;

  while l < r, do

  while l < right & a[l] < pivot , set l = l + 1

  while r > left & a[r] >= pivot , set r = r - 1

  if l < r, swap a[l] and a[r]

  Set a[left] = a[r], a[r] = pivot

  Terminate

Example of partitioning

- choose pivot: 4 3 6 9 2 4 3 1 2 1 8 9 3 5 6
- search: 4 3 6 9 2 4 3 1 2 1 8 9 3 5 6
- swap: 4 3 3 9 2 4 3 1 2 1 8 9 6 5 6
- search: 4 3 3 9 2 4 3 1 2 1 8 9 6 5 6
- swap: 4 3 3 1 2 4 3 1 2 9 8 9 6 5 6
- search: 4 3 3 1 2 4 3 1 2 9 8 9 6 5 6
- swap: 4 3 3 1 2 2 3 1 4 9 8 9 6 5 6
- search: 4 3 3 1 2 2 3 1 4 9 8 9 6 5 6
- swap with pivot: 1 3 3 1 2 2 3 4 4 9 8 9 6 5 6

Partition Implementation (Java)

```
static int Partition(int[] a, int left, int right) {
   int p = a[left], l = left + 1, r = right;
   while (l < r) {
      while (l < right && a[l] < p)   l++;
      while (r > left && a[r] >= p)   r--;
      if (l < r) {
         int temp = a[l]; a[l] = a[r]; a[r] = temp;
      }
   }
   a[left] = a[r];
   a[r] = p;
   return r;
}
```

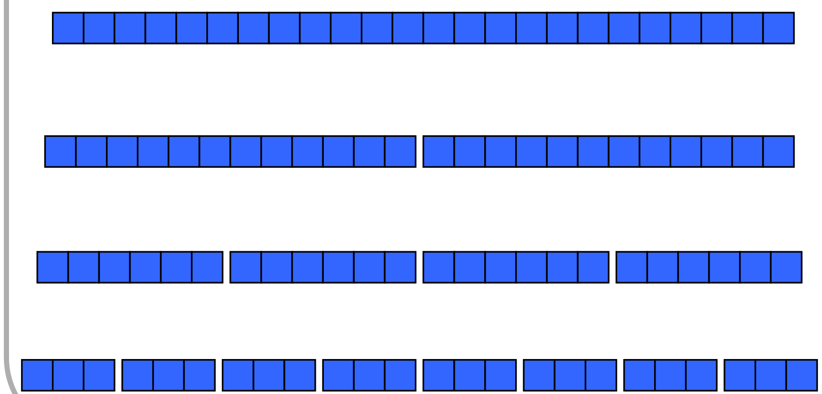
Quicksort Implementation (Java)

```
static void Quicksort(int[] array, int left, int right)
{
   if (left < right) {
      int p = Partition(array, left, right);
      Quicksort(array, left, p - 1);
      Quicksort(array, p + 1, right);
   }
}
```

Analysis of quicksort—best case

- Suppose each partition operation divides the array almost exactly in half
- Then the depth of the recursion in $\log_2 n$
  - Because that's how many times we can halve n
- We note that
  - Each partition is linear over its subarray
  - All the partitions at one level cover the array

Partitioning at various levels



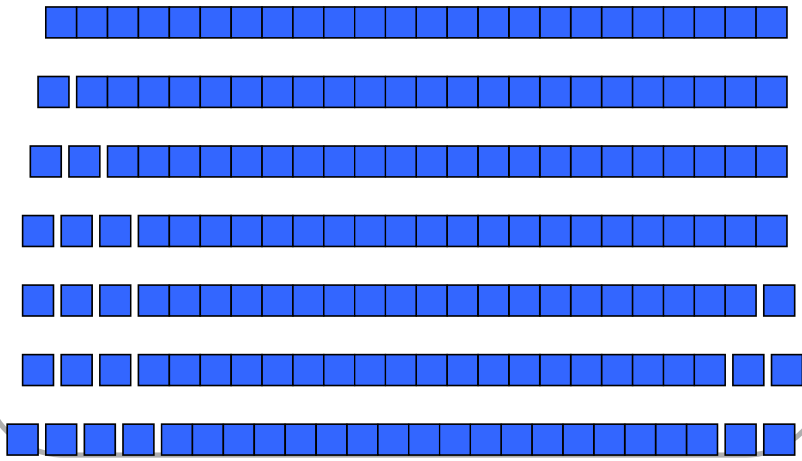
Best Case Analysis

- We cut the array size in half each time
- So the depth of the recursion in $\log_2 n$
- At each level of the recursion, all the partitions at that level do work that is linear in n
- $O(\log_2 n) * O(n) = O(n \log_2 n)$
- Hence in the best case, quicksort has time complexity $O(n \log_2 n)$
- What about the worst case?

Worst case

- In the worst case, partitioning always divides the size n array into these three parts:
  - A length one part, containing the pivot itself
  - A length zero part, and
  - A length $n-1$ part, containing everything else
- We don't recur on the zero-length part
- Recurring on the length $n-1$ part requires (in the worst case) recurring to depth $n-1$

Worst case partitioning



Worst case for quicksort

- In the worst case, recursion may be n levels deep (for an array of size n)
- But the partitioning work done at each level is still n
- O(n) * O(n) = $O(n^2)$
- So worst case for Quicksort is $O(n^2)$
- When does this happen?
  - There are many arrangements that *could* make this happen
  - Here are two common cases:
    - When the array is already sorted
    - When the array is *inversely* sorted (sorted in the opposite order)

Typical case for quicksort

- If the array is sorted to begin with, Quicksort is terrible: $O(n^2)$
- It is possible to construct other bad cases
- However, Quicksort is *usually* $O(n \log_2 n)$
- The constants are so good that Quicksort is generally the faster algorithm.
- Most real-world sorting is done by Quicksort

Picking a better pivot

- Before, we picked the *first* element of the subarray to use as a pivot
  - If the array is already sorted, this results in $O(n^2)$ behavior
  - It's no better if we pick the *last* element
- We could do an *optimal* quicksort (guaranteed O(n log n)) if we always picked a pivot value that exactly cuts the array in half
  - Such a value is called a **median**: half of the values in the array are larger, half are smaller
  - The easiest way to find the median is to *sort* the array and pick the value in the middle (!)

Median of three

- Obviously, it doesn’t make sense to sort the array in order to find the median to use as a pivot.
- Instead, compare just *three* elements of our (sub)array—the first, the last, and the middle
  - Take the *median* (middle value) of these three as the pivot
  - It’s possible (but not easy) to construct cases which will make this technique $O(n^2)$

Quicksort for Small Arrays

- For very small arrays (N<= 20), quicksort does not perform as well as insertion sort
- A good cutoff range is N=10
- Switching to insertion sort for small arrays can save about 15% in the running time

Mergesort vs Quicksort

- Both run in O(*n* lg*n*)
- Compared with Quicksort, Mergesort has less number of comparisons but larger number of moving elements
- In Java, an element comparison is expensive but moving elements is cheap. Therefore, Mergesort is used in the standard Java library for generic sorting

Mergesort vs Quicksort

In C++, copying objects can be expensive while comparing objects often is relatively cheap. Therefore, quicksort is the sorting routine commonly used in C++ libraries