Mathematics for Computer Graphics

CSE 581 – Roger Crawfis
(slides developed from Korea University slides)

Spaces

- Scalars
- (Linear) Vector Space
  - Scalars and vectors
- Affine Space
  - Scalars, vectors, and points
- Euclidean Space
  - Scalars, vectors, points
  - Concept of distance
- Projections

Scalars (1/2)

- **Scalar Field**
  - Ex) Ordinary real numbers and operations on them
- Two Fundamental Operations
  - **Addition** and **multiplication**
    \[
    \forall \alpha, \beta \in S, \quad \alpha + \beta \in S, \quad \alpha \cdot \beta \in S
    \]
    \[
    \alpha + \beta = \beta + \alpha \quad \text{Commutative}
    \]
    \[
    \alpha \cdot \beta = \beta \cdot \alpha \quad \text{Commutative}
    \]
    \[
    \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \quad \text{Associative}
    \]
    \[
    \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma \quad \text{Associative}
    \]
    \[
    \alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma) \quad \text{Distributive}
    \]

Scalars (2/2)

- Two Special Scalars
  - **Additive identity**: 0, **multiplicative identity**: 1
    \[
    \alpha + 0 = 0 + \alpha = \alpha
    \]
    \[
    \alpha \cdot 1 = 1 \cdot \alpha = \alpha
    \]
  - **Additive inverse** $-\alpha$ and **multiplicative inverse** $\alpha^{-1}$ of $\alpha$
    \[
    \alpha + (-\alpha) = 0
    \]
    \[
    \alpha \cdot \alpha^{-1} = 1
    \]
Vector Spaces (1/4)

- Two Entities: **Scalars** and **Vectors**
- Vectors
  - Directed line segments
  - $n$-tuples of numbers
  - Two operations: vector-vector addition, scalar-vector multiplication
- Special Vector: **Zero Vector**
  \[ u + 0 = u \]
  \[ u + (-u) = 0 \]
- Let $u$ denote a vector

Vector Spaces (2/4)

- Scalar-Vector Multiplication
  \[ \alpha (u + v) = \alpha u + \alpha v \]
  \[ (\alpha + \beta)u = \alpha u + \beta u \]
- $u$ and $v$: vectors, $\alpha$ and $\beta$: scalars

Vector Spaces (3/4)

- Vectors $= n$-tuples $v = (v_1, v_2, \ldots, v_n)$
- Vector-vector addition
  \[ u + v = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \]
- Scalar-vector multiplication
  \[ \alpha v = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n) \]
- Vector space: $\mathbb{R}^n$

Vector Spaces (4/4)

- **Dimension**
  - The greatest number of linearly independent vectors
- **Basis**
  - $n$ linearly independent vectors ($n$: dimension)
- Representation $\{\beta_i\}$
  - Unique expression in terms of the basis vectors
  \[ \vec{v} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \cdots + \beta_n \vec{v}_n \]
- **Change of Basis**: Matrix $M$
  - Other basis $\vec{v}_1', \vec{v}_2', \ldots, \vec{v}_n'$
  \[ \vec{v} = \beta_1' \vec{v}_1' + \beta_2' \vec{v}_2' + \cdots + \beta_n' \vec{v}_n' \]
  \[ \begin{bmatrix} \beta_1' \\ \beta_2' \\ \vdots \\ \beta_n' \end{bmatrix} = M \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \]
**Affine Spaces (1/2)**

- **No Geometric Concept in Vector Space!!**
  - Ex) location and distance
  - Vectors: magnitude and direction, no position
- **Coordinate System**
  - **Origin**: a particular reference point

**Affine Spaces (2/2)**

- **Third Type of Entity**: Points
- **New Operation**: Point-Point Subtraction
  - \( P \) and \( Q \): any two points
    \[ \vec{v} = P - Q \]
  - \( \vec{v} \) = vector-point addition
    \[ \vec{P} = \vec{v} + Q \]
  - Frame: a Point \( P_0 \) and a Set of Vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \)
    - Representations of the vector and point: \( n \) scalars
      - **Vector**: \( v = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n \)
      - **Point**: \( P = P_0 + \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \cdots + \beta_n \vec{v}_n \)

**Euclidean Spaces (1/2)**

- **No Concept of How Far Apart Two Points in Affine Spaces!!**
- **New Operation**: Inner (dot) Product
  - Combine two vectors to form a real
    - \( \alpha, \beta, \gamma, \ldots \): scalars, \( u, v, w, \ldots \): vectors
      \[ u \cdot v = v \cdot u \]
      \[ (\alpha u + \beta v) \cdot w = \alpha u \cdot w + \beta v \cdot w \]
      \[ v \cdot v > 0 \quad \text{if} \quad v \neq 0 \]
      \[ 0 \cdot 0 = 0 \]
  - **Orthogonal**: \( u \cdot v = 0 \)

**Euclidean Spaces (2/2)**

- **Magnitude (length)** of a vector
  \[ |v| = \sqrt{v \cdot v} \]
- **Distance** between two points
  \[ |P - Q| = \sqrt{(P - Q) \cdot (P - Q)} \]
- **Measure of the angle between two vectors**
  \[ u \cdot v = |u||v| \cos \theta \]
  - \( \cos \theta = 0 \Rightarrow \) orthogonal
  - \( \cos \theta = 1 \Rightarrow \) parallel
**Projections**

- Problem of Finding the Shortest Distance from a Point to a Line of Plane
- Given Two Vectors,
  - Divide one into two parts: one parallel and one orthogonal to the other
  
  \[
  w = \alpha v + u \quad \text{Projection of one vector onto another}
  \]
  \[
  \begin{align*}
  w \cdot v &= \alpha v \cdot v + u \cdot v = \alpha v \cdot v \\
  \therefore \alpha &= \frac{w \cdot v}{v \cdot v} \\
  \therefore u &= w - \alpha v = w - \frac{w \cdot v}{v \cdot v}
  \end{align*}
  \]

**Matrices**

- Definitions
- Matrix Operations
- Row and Column Matrices
- Rank
- Change of Representation
- Cross Product

**What is a Matrix?**

- A matrix is a set of elements, organized into rows and columns

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\]

**Definitions**

- \(n \times m\) Array of Scalars (\(n\) Rows and \(m\) Columns)
  - \(n\): row dimension of a matrix, \(m\): column dimension
  - \(m = n \Rightarrow \text{square matrix}\) of dimension \(n\)
  - Element \(\{a_{ij}\}, \; i = 1, \ldots, n, \; j = 1, \ldots, m\)
  - **Transpose**: interchanging the rows and columns of a matrix
    \[
    A^T = [a_{ji}]
    \]
  - Column Matrices and Row Matrices
    - **Column matrix** \((n \times 1\) matrix): \(b = [b_i] = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\)
    - **Row matrix** \((1 \times n\) matrix): \(b^T\)
Basic Operations

- Addition, Subtraction, Multiplication

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
+ 
\begin{bmatrix}
  e & f \\
  g & h
\end{bmatrix}
= 
\begin{bmatrix}
  a+e & b+f \\
  c+g & d+h
\end{bmatrix}
\] 
Just add elements

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
- 
\begin{bmatrix}
  e & f \\
  g & h
\end{bmatrix}
= 
\begin{bmatrix}
  a-e & b-f \\
  c-g & d-h
\end{bmatrix}
\] 
Just subtract elements

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\cdot 
\begin{bmatrix}
  e & f \\
  g & h
\end{bmatrix}
= 
\begin{bmatrix}
  ae+bg & af+bh \\
  ce+dg & cf+dh
\end{bmatrix}
\] 
Multiply each row by each column

Matrix Operations (1/2)

- **Scalar-Matrix Multiplication**
  \[\alpha A = [\alpha a_{ij}]\]

- **Matrix-Matrix Addition**
  \[C = A + B = [a_{ij} + b_{ij}]\]

- **Matrix-Matrix Multiplication**
  - \[A: n \times l \text{ matrix}, B: l \times m \rightarrow C: n \times m \text{ matrix}\]
  \[C = AB = [c_{ij}]\]
  \[c_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj}\]

Matrix Operations (2/2)

- **Properties of Scalar-Matrix Multiplication**
  \[\alpha (\beta A) = (\alpha \beta)A\]
  \[\alpha \beta A = \beta \alpha A\]

- **Properties of Matrix-Matrix Addition**
  - Commutative: \[A + B = B + A\]
  - Associative: \[A + (B + C) = (A + B) + C\]

- **Properties of Matrix-Matrix Multiplication**
  \[A(BC) = (AB)C\]
  \[AB \neq BA\]

- **Identity Matrix** \(I\) (Square Matrix)

\[I = \begin{bmatrix} a_{ij} \end{bmatrix}\]
\[a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}\]

\[AI = A\]
\[IB = B\]

Multiplication

- **Is AB = BA?** Maybe, but maybe not!

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  e & f \\
  g & h
\end{bmatrix}
= 
\begin{bmatrix}
  ea+bg & \cdot & \cdots \\
  ea+fc & \cdot & \cdots
\end{bmatrix}
\]

- **Heads up:** multiplication is NOT commutative!
**Row and Column Matrices**

- Column Matrix
  \[
  \mathbf{p} = \begin{bmatrix}
  x \\
  y \\
  z
  \end{bmatrix}
  \]
- Column Matrix \(\mathbf{p}^T\): row matrix
- **Concatenations**
  - Associative
    \[
    \mathbf{p}' = \mathbf{A}\mathbf{p}
    \]
  - By Row Matrix
    \[
    (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T
    \]
    \[
    \mathbf{p}'^T = \mathbf{p}^T\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T
    \]

**Vector Operations**

- Vector: 1 x N matrix
- Interpretation: a line in N dimensional space
- Dot Product, Cross Product, and Magnitude defined on vectors only

**Vector Interpretation**

- Think of a vector as a line in 2D or 3D
- Think of a matrix as a transformation on a line or set of lines

\[
\begin{bmatrix}
  x \\
  y
  \end{bmatrix}
\begin{bmatrix}
  a & b \\
  c & d
  \end{bmatrix}
= \begin{bmatrix}
  x' \\
  y'
  \end{bmatrix}
\]

**Vectors: Dot Product**

- Interpretation: the dot product measures to what degree two vectors are aligned

\[
\mathbf{A} + \mathbf{B} = \mathbf{C}
\]

(use the head-to-tail method to combine vectors)
**Vectors: Dot Product**

\[ a \cdot b = ab^T = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = ad + be + cf \]

Think of the dot product as a matrix multiplication.

\[ \|a\|^2 = a a^T = \sqrt{aa + bb + cc} \]

The magnitude is the dot product of a vector with itself.

\[ a \cdot b = \|a\|\|b\|\cos(\theta) \]

The dot product is also related to the angle between the two vectors – but it doesn’t tell us the angle.

**Vectors: Cross Product**

- The cross product of vectors A and B is a vector C which is perpendicular to A and B.
- The magnitude of C is proportional to the cosine of the angle between A and B.
- The direction of C follows the right hand rule – this why we call it a “right-handed coordinate system.”

\[ \|a \times b\| = \|a\|\|b\|\sin(\theta) \]

**Inverse of a Matrix**

- Identity matrix: \( AI = A \)
- Some matrices have an inverse, such that: \( AA^{-1} = I \)
- Inversion is tricky: \((ABC)^{-1} = C^{-1}B^{-1}A^{-1}\)

Derived from non-commutativity property.

**Determinant of a Matrix**

- Used for inversion.
- If \( \text{det}(A) = 0 \), then A has no inverse.
- Can be found using factorials, pivots, and cofactors!
- Lots of interpretations – for more info, take 18.06.
**Determinant of a Matrix**

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg
\]

Sum from left to right
Subtract from right to left!
Note: \( N! \) terms

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**Change of Representation**

- **Matrix Representation of the Change between the Two Bases**
  - Ex) two bases \( \{u_1, u_2, \ldots, u_n\} \) and \( \{v_1, v_2, \ldots, v_n\} \)
    
    \[
    v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n
    \]
    
    or
    
    \[
    v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n
    \]

- **Representations of** \( v \)

  \[
  a = [\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n]^T \quad \text{or} \quad b = [\beta_1 \ \beta_2 \ \ldots \ \beta_n]^T
  \]

- **Expression of** \( \{u_1, u_2, \ldots, u_n\} \) in the basis \( \{v_1, v_2, \ldots, v_n\} \)
  
  \[
  u_i = \gamma_{i1} v_1 + \gamma_{i2} v_2 + \cdots + \gamma_{in} v_n, \quad i = 1, \ldots, n
  \]

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**Cross Product**

- **In 3D Space, a unit Vector,** \( w \), **is Orthogonal to Given Two Nonparallel Vectors,** \( u \) and \( v \)

  \[
  w \cdot u = w \cdot v = 0
  \]

- **Definition**

  \[
  w = u \times v = \begin{bmatrix}
  \alpha_2 \beta_3 - \alpha_3 \beta_2 \\
  \alpha_3 \beta_1 - \alpha_1 \beta_3 \\
  \alpha_1 \beta_2 - \alpha_2 \beta_1
  \end{bmatrix}
  \]

  where \( u = (\alpha_1, \alpha_2, \alpha_3) \), \( v = (\beta_1, \beta_2, \beta_3) \)

- **Consistent Orientation**

  - Ex) \( x \)-axis \( \times \) \( y \)-axis = \( z \)-axis