1. (Propositional Calculus – 10 points)
---------------------------------------
Let P, Q, R range over state predicates of some program.
Prove or disprove the following:

a) \( P \lor (P \land Q) \equiv P \)

b) \( P \land (Q \lor R) \equiv (P \lor Q) \land (P \lor R) \)

c) \( \neg(P \equiv Q) \equiv \neg P \equiv \neg Q \)

d) \( P \equiv Q \equiv (P \lor Q) \equiv (P \land Q) \)

2. (More Propositional Calculus – 6 points)
------------------------------------------

a) Prove \( \neg \neg P \equiv P \)

b) Prove the identity of \( \lor \), \( P \lor \text{false} \equiv P \), by transforming its more structured side into its simpler side.

c) Prove \( P \Rightarrow Q \equiv \neg P \lor \neg Q \)

3. (Predicate Calculus – 10 points)
---------------------------------

a) Prove \( (\forall x : R : P \equiv Q) \Rightarrow ((\forall x : R : P) \equiv (\forall x : R : Q)) \)

b) Prove \( \neg(\exists x : R : P) \equiv (\forall x : R : \neg P) \)

c) Translate the following English statements into predicate logic:
   (i) Every positive integer is smaller than the absolute value of some negative integer. (Use \( \text{abs.i} \) for the absolute value of \( i \))
   (ii) Real number \( i \) is the largest real solution of the equation \( f.i = i + 1 \)
   (iii) No integer is larger than all others.

d) Translate into English the meaning of:
   (i) \( (\exists x, y : x \in R \land y \in R : (f.x < 0 \land 0 < f.y) \Rightarrow (\exists z : z \in \text{Reals} : f.z = 0)) \)
   (ii) \( (\forall z : z \in \text{Integers} \land \text{even}.z : (\forall w : w \in \text{Integers} \land \text{odd}.w : z \neq w)) \)

4. (Closure) -- 30 points
-------------------------
Let P and Q range over state predicates of a program \( \text{prog} \). Recall that the statements of each action of \( \text{prog} \) are terminating.
Recall that in class we defined:

\[
\text{closed } P \text{ iff } \{P\} \ \text{prog} \ \{P\}
\]
True or False? (Explain your answer.)

a) closed false
b) closed true
c) (closed $P$ or closed $Q$) implies (closed $(P \lor Q)$)
d) (closed $\neg P$) implies (closed $P$)
e) (closed $(P \lor Q)$) implies ($\forall s :: \{P\} s \{Q\}$)
f) (exists $s :: \{P\} s \{false\}$) implies (closed $\neg P$)
g) closed $(P \lor Q)$ implies ($\forall s :: \{P\} s \{P \lor Q\}$)
h) closed $P$ and closed $Q$ and $(R \Rightarrow (P \land Q))$ implies closed $R$
i) closed $P$ and closed $Q$ and closed $R$
implies closed $(P \land (Q \land R))$

5. (Leads-to) -- 24 points
--------------------------

Let $P$, $Q$, and $R$ range over state predicates of a program $prog$.

True or False? (Explain your answer.)

a) $false$ leads-to $P \lor Q$
b) $(P$ leads-to $Q)$ implies ($(P \land Q)$ leads-to $Q$)
c) $(P$ leads-to $Q)$ implies ($(P \land R)$ leads-to $Q$)
d) ($(P$ leads-to $Q)$ and $(P$ leads-to $R)$) implies $(P$ leads-to $(Q \land R)$)
e) $(P$ leads-to $Q)$ and ($(Q \lor R)$ leads-to $T)$
implies $(P \lor R)$ leads-to $T$)
f) $P$ leads-to $Q$ and $P$ leads-to $R$ and closed $R$
implies $(P$ leads-to $(Q \land R)$)

6. (Variant functions) - 20 points
----------------------------------

For each program described below, prove, by exhibiting a variant function, that the desired progress property holds, or show that the progress property does not hold. Assume the semantics of minimal progress: At every step in the computation, if some action is enabled, then some enabled action is executed.
a) Let $x.j$ be an integer for $0 \leq j < N$. For each $j$ in the range $0 < j < N$, consider the program action:

$$x.j < x.(j - 1) \rightarrow x.j, x.(j - 1) := x.(j - 1), x.j$$

The progress property to be verified is:

true leads-to $(\forall j : 0 < j < N : x.j \geq x.(j - 1))$

b) Given are line segments $L.1, L.2, ..., L.N$ in the X-Y plane (assume all $2N$ endpoints are unique) and a program that consists of one action for each pair $(L.j, L.k)$ of line segments:

$L.j$ and $L.k$ intersect $\rightarrow$ swap any one endpoint of $L.j$ with any one endpoint of $L.k$, thus making $L.j$ and $L.k$ nonintersecting

The progress property to be verified is: “the program eventually terminates”

7. (Verifying closure and leads-to)

Consider the program $TRANS$ over the boolean variables $b$, $c$, and $d$:

$$
\begin{align*}
  b \quad \rightarrow & \quad c := true \\
  b \land c \quad \rightarrow & \quad d := true
\end{align*}
$$

Are the following properties true in $TRANS$? (Explain your answer carefully. A formal proof is not necessary.)

(i) closed (¬$b \land c$)
(ii) closed (¬$c \land d$)
(iii) $c$ leads-to $d$
(iv) $b$ leads-to $d$

Does the variant function

$(3 - \text{number of variables of } TRANS \text{ that are true})$

suffice to verify the leads-to predicate in part (iii)? in part (iv)?
8. (Distributed load balancing)  
-----------------------------------

Prove either that the desired liveness specification holds
by exhibiting a variant function, or show that it does not hold.

Let \( x.j \) be an integer for each node \( j \) in an undirected graph.
For each pair of neighboring nodes \( j \) and \( k \) in the graph,
consider the program action:

\[
(x.j - x.k) > 1 \quad \rightarrow \quad x.j, x.k := x.j - 1, x.k + 1
\]

The liveness specification to be verified for this set of actions is:

\[
\text{true leads-to (forall j, k: j and k are neighboring nodes:} \quad |x.j-x.k| \leq 1)\]

9. (Verifying Hoare-triples)

Let $m$, $n$, and $l$ be integers, and $M$ and $N$ be integer constants. Carefully prove or disprove the following Hoare-triples. (Formal proofs are not necessary, but are encouraged).

(a) \[ \{ m = M \} \]
\[ m < 0 \rightarrow m := -m \]
\[ \{ m = |M| \} \]

(b) \[ \{ m > M \} \]
\[ m > n \rightarrow m, n := n, m \]
\[ \{ m \leq n \} \]

Here are two new rules about Hoare-triples:

**Rule of Sequential Assignment:**
Let $x$ and $y$ be variables and $E$ and $F$ be expressions whose value are in the domain of $x$ and $y$, respectively, and let $P$ be a state predicate.

\[ \{ (P [y := F]) [x := E] \} \ true \rightarrow x := E ; y := F \ \{ P \} \]

**Rule of Guards:**
Let $prog$ be a program with two actions $g_1 \rightarrow st_1$ and $g_2 \rightarrow st_2$, and let $Q$ and $R$ be state predicates of $prog$.

\[ Q \Rightarrow g_1 \lor g_2 , \]
\[ \{ Q \} g_1 \rightarrow st_1 \ \{ R \} , \]
\[ \{ Q \} g_2 \rightarrow st_2 \ \{ R \} \]

implies

\[ \{ Q \} prog \ \{ R \} \]

Prove or disprove the following Hoare-triples:

(c) \[ \{ m = M \land n = N \} \]
\[ true \rightarrow n := n + m ; m := n - m ; n := n - m \]
\[ \{ m = N \land n = M \} \]

(d) \[ \{ true \} \ l \leq m \rightarrow n := m \ \ |
\[ m \leq l \rightarrow n := l \ \ \{ n = \max(l, m) \} \]