On Computing Handle and Tunnel Loops

Tamal K. Dey   Kuiyu Li   Jian Sun

Department of Computer Science and Engineering
The Ohio State University
Columbus, OH, 43210, USA

Abstract

Many applications seek to identify features like 'handles' and 'tunnels' in a shape bordered by a surface embedded in three dimensions. To this end we define handle and tunnel loops on surfaces which can help identifying these features. We show that a closed surface of genus g always has g handle and g tunnel loops induced by the embedding. For a class of shapes that retract to graphs, we characterize these loops by a linking condition with these graphs. These characterizations lead to algorithms for detection and generation of these loops. We provide an implementation with applications to feature detection and topology simplification to show the effectiveness of the method.

1 Introduction

Many applications need to identify features such as 'handles' and 'tunnels' induced by the embedding of a connected closed surface in three dimensions. We define a class of loops on M, called handle and tunnel loops, that help identifying these handles and tunnels respectively. Intuitively, a loop is a handle if it spans a disk (surface) in the bounded space bordered by M. If one cuts M along such a loop and fills the boundary with that disk, one eliminates a handle. Similarly, a tunnel loop spans a disk (surface) in the unbounded space bordered by M and its removal eliminates a tunnel. Figure 1 shows three handle and three tunnel loops on a CAD surface. In this paper we provide a formal definition of handle and tunnel loops in terms of homology groups and provide topological analyses that lead to the algorithms for their detection and generation. Our algorithm can be a basis for applications that require to recognize features such as handles in a shape and tunnels in its complement or to simplify a shape topologically by eliminating insignificant handles and tunnels [BK05, ESV97, GW01, WHDS04, ZJH07].

Researchers have looked into the problem of computing nontrivial loops on surfaces with various conditions. Vegter and Yap [VY90] and Dey and Schipper [DS95] gave linear time algorithms to compute polygonal schemas whose removal cuts the surface into a disk. Erickson and Har-Peled [EHP04] showed that computing graphs of shortest length whose removal cut the surface into a disk is NP-hard. Verdière and Lazarus [ECdVL05] gave an algorithm for computing a system of loops on a surface which is shortest among the homotopy class of a given system. Erickson and Whittlesey [EW05] gave a greedy algorithm to compute the shortest system of loops, among all systems of loops, relaxing the homotopy condition.

The above works were mainly concerned with computing a set of non trivial loops while optimizing some metric on the surface. Our goal is different. We seek to compute only specific loops that are handles and/or tunnels. One fundamental difference is that the aforementioned works do not take into account the embedding $M \to \mathbb{R}^3$ whereas handle and tunnel loops become meaningful only for embedded surfaces $M \subset \mathbb{R}^3$.

We formalize the ideas of handle and tunnel loops and provide an existence proof for them. We argue that the notion of handle and tunnel loops loses its intuitive meaning if the surface has a knotted embedding. We define *graph retractable* surfaces that avoid these knotted embeddings. These are surfaces whose interiors and exteriors deformation retract to graphs called *core graphs*. We present algorithms to detect and generate handle and tunnel loops on such surfaces. Specifically, the main contributions of this paper are:

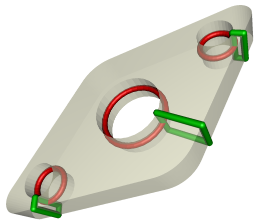


Figure 1: Cad gadget: the red loops represent tunnels and the green ones represent handles.

DEFINITION AND EXISTENCE. We provide a formal definition of handle and tunnel loops and prove their existence.
DETECTION. We characterize handle and tunnel loops on graph retractable surfaces in terms of their linking with the core graphs. This leads to an algorithm for detecting handle and tunnel loops.
GENERATION. We show that there exist a special class of handle and tunnel loops that link minimally with the core graphs and present an algorithm to compute them.
IMPLEMENTATION. We present an implementation of our algorithm which incorporates the geometry of the surface more intimately. The results of our implementation show that the method is effective in practice.
APPLICATION. We apply our algorithm to the problem of computing 'handle' and 'tunnel' features in shapes which can further be used for topology simplification. Again, the results show the effectiveness of the method.

2 Preliminaries

We state some standard concepts from topology. For a more detailed introduction, the interested readers may consult Munkres [Mun93] or Hatcher [Hat02].

Let X be any topological space. A singular k-simplex is defined as a continuous map $\sigma : \Delta^k \to X$ where Δ^k is the standard k-simplex which is the convex hull of $\{e_i\}_{i=1}^{k+1}$. Each e_i is a vector in $\mathbb{R}^{k+1}$ and its jth component is δ_{ij} where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. A *k-chain* is a finite linear combination of singular k-simplices. In this paper we assume the coefficient ring to be $\mathbb{Z}$, the set of integers. The set of all k-chains forms a chain group $C_k(X)$ under addition. The boundary operator ∂ of a singular k-simplex σ takes σ to a collection of maps that are the restriction of σ on the boundary facets of Δ^k, which form a $(k-1)$-chain. A boundary operator $\partial : C_k(X) \to C_{k-1}(X)$ can be defined by the linear extension. A k-chain is called a k-cycle if its boundary is empty and a k-boundary if it is the boundary of a $(k+1)$-chain. A k-boundary must be a k-cycle. Let Z_k and B_k denote the set of all k-cycles and k-boundaries respectively. Both Z_k and B_k are subgroups of $C_k(X)$. The kth *homology group* of X, denoted $H_k(X)$, is the quotient group Z_k/B_k. Since $C_K(X)$ is abelian, so is $H_k(X)$. If X is a simplicial complex, one could define the simplicial homology group for X where each k-simplex plays role of a singular k-simplex. It turns out that these two homology groups are the same up to isomorphism.

In this paper, we are interested in the first homology group $H_1(X)$ which is a set of equivalent classes of loops defined as continuous maps from circles into X. Each such class called *homology class* can be represented by a loop in X if X is path connected. We let $[l]$ denote the homology class represented by a loop l. The topological spaces we consider in this paper are all compact subspaces of $\mathbb{R}^3$. Their first homology groups are free abelian groups and hence have a basis where every element of the group can be written uniquely as a finite linear combination of elements in the basis. The elements in a basis are also referred as *generators*.

A topological space X *deformation retracts* to its subspace A if there is a continuous map $F : X \times [0,1] \to$

X satisfying the following conditions

- $F(x,0) = x$ for any $x \in X$
- $F(x,1) \in A$ for any $x \in X$
- $F(a,t) = a$ for any $a \in A$ and any $t \in [0,1]$.

X and A are of the same homotopy type if X deformation retracts to A. We also use the concepts of cohomology, Mayer-Vietoris sequences, and others whose excellent expositions can be found in Hatcher [Hat02].

3 Definition and existence

Let M be a connected, closed (compact and without boundary), and orientable surface. The genus g of M is the maximum number of disjoint simple loops whose removal does not disconnect M. Two closed, connected and orientable surfaces are homeomorphic if and only if they have the same genus. To make our argument simple, let M sit inside a three sphere $\mathbb{S}^3$, which is the compactification of $\mathbb{R}^3$. Being embedded in $\mathbb{S}^3$, the surface M has to be orientable. It separates $\mathbb{S}^3$ into two parts. Given an orientation of M, we may designate a connected component, say I, of $\mathbb{S}^3 \setminus M$ as *inside* and the other, say O, as *outside*. Let

$$\mathbb{I} = I \cup M \text{ and } \mathbb{O} = O \cup M.$$

Notice that both $\mathbb{I}$ and $\mathbb{O}$ have M on their boundary. As a compact orientable surface in $\mathbb{S}^3$, M admits an open tubular neighborhood in $\mathbb{S}^3$, denoted Σ.

Definition 1 *A loop on M is a* tunnel loop *if the homology class carried by it is trivial in $H_1(\mathbb{O})$ and non trivial in $H_1(\mathbb{I})$.*

Definition 2 *A loop on M is a* handle loop *if the homology class carried by it is trivial in $H_1(\mathbb{I})$ and non trivial in $H_1(\mathbb{O})$.*

By definition, set of tunnel loops are disjoint from set of handle loops, namely no loop on M can be both handle and tunnel. In addition, a tunnel loop or a handle loop must be non trivial in $H_1(M)$. This is because the inclusion map from M to $\mathbb{I}$ induces a homomorphism from $H_1(M)$ to $H_1(\mathbb{I})$. Similarly the inclusion map from M to $\mathbb{O}$ induces a homomorphism from $H_1(M)$ to $H_1(\mathbb{O})$. Hence a loop on M that represents the trivial element in $H_1(M)$ remains representing the trivial element both in $H_1(\mathbb{I})$ and $H_1(\mathbb{O})$. However, not every non trivial loop on M is either handle or tunnel. For example, the loop shown in Figure 2 is neither a handle nor a tunnel since it is non trivial in both $H_1(\mathbb{I})$ and $H_1(\mathbb{O})$.

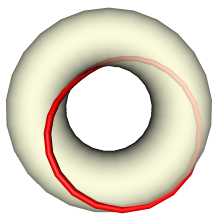


Figure 2: The loop on the torus is neither a handle nor a tunnel.

It is not immediately obvious if a connected closed surface of genus g admits g handle and g tunnel loops. In particular, 'knotted' embedding of surfaces makes it a non trivial fact as standard surface classification theorem cannot be applied to derive it.

Theorem 1 *For any connected closed surface $M \subset \mathbb{S}^3$ of genus g, there exist g handle loops $\{h_i\}_{i=1}^g$ forming a basis for $H_1(\mathbb{O})$ and g tunnel loops $\{t_i\}_{i=1}^g$ forming a basis for $H_1(\mathbb{I})$. Furthermore, $\{[h_i]\}_{i=1}^g$ and $\{[t_i]\}_{i=1}^g$ form a basis for $H_1(M)$.*

Proof 1 *The tubular neighborhood Σ of M in $\mathbb{S}^3$ satisfies*

$$\begin{aligned} \mathbb{S}^3 &= (\mathbb{I} \cup \Sigma) \cup (\mathbb{O} \cup \Sigma) \text{ and} \\ \Sigma &= (\mathbb{I} \cup \Sigma) \cap (\mathbb{O} \cup \Sigma). \end{aligned}$$

Since both $\mathbb{I} \cup \Sigma$ and $\mathbb{O} \cup \Sigma$ are open, we have the following Mayer-Vietors sequence, which is exact.

$$\begin{aligned} \cdots \to H_2(\mathbb{S}^3) \xrightarrow{\alpha} H_1(\Sigma) \xrightarrow{\beta} H_1(\mathbb{I} \cup \Sigma) \oplus H_1(\mathbb{O} \cup \Sigma) \\ \xrightarrow{\gamma} H_1(\mathbb{S}^3) \to \cdots \end{aligned}$$

Since both $H_2(\mathbb{S}^3)$ and $H_1(\mathbb{S}^3)$ are trivial, β is an isomorphism induced by the inclusion map. Since Σ, $\mathbb{I} \cup \Sigma$, and $\mathbb{O} \cup \Sigma$ deformation retract to M, $\mathbb{I}$, and $\mathbb{O}$ respectively, there is an isomorphism from $H_1(M)$ to $H_1(\mathbb{I}) \oplus H_1(\mathbb{O})$, which is also induced by inclusion. It follows that

$$rank(H_1(\mathbb{I})) + rank(H_1(\mathbb{O})) = rank(H_1(M)) = 2g. \tag{1}$$

Also, it is known that (Theorem 19 in [Moi77] p. 172)

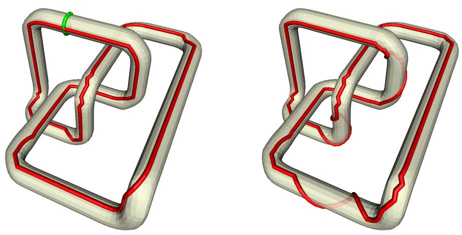
$$rank(H_1(\mathbb{I})) \geq g \text{ and } rank(H_1(\mathbb{O})) \geq g. \tag{2}$$

Equations 1 and 2 force both the ranks of $H_1(\mathbb{I})$ and $H_1(\mathbb{O})$ equal g. Now one can choose g elements $(c_i, 0)$, $i = 1, .., g$, from a basis in $H_1(\mathbb{I}) \oplus H_1(\mathbb{O})$ whose preimage by the inclusion isomorphism provides g elements in $H_1(M)$. These g elements, by definition, are classes of g tunnel loops forming a basis of $H_1(\mathbb{I})$. Similar arguments show existence of g handle loops.

4 Graph retractable surface

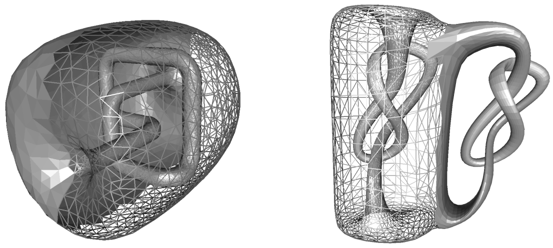
Although Theorem 1 assures the existence of g handle loops and g tunnel loops on all connected closed surfaces in $\mathbb{S}^3$, they do not bear intuitive meaning of handles and tunnels in 'knotted' surfaces. Figure 3 shows such a surface. It is obtained by thickening a trefoil knot, namely M is the boundary of the product, $K \times D$, of a trefoil K and a 2-disk D. The red loop in Figure 3(a) is obtained by projecting the trefoil knot on the surface M. In contrary to the natural intuition this loop is not a tunnel loop. It is not trivial in $H_1(\mathbb{O})$ though it generates $H_1(\mathbb{I})$. Actually one can derive that its homology class in $H_1(\mathbb{O})$ is 3 times the generator (the green loop in Figure 3(a)), which explains why the tunnel loop shown in Figure 3(b) contains three windings.

We want to avoid such "knotted" surfaces by considering the property that at least one of the spaces $\mathbb{I}$ and $\mathbb{O}$ does not deformation retract to a graph. For example, take the above thickened trefoil which can be considered as a knotted torus. The fundamental group of the complement of $K \times D$ in $\mathbb{S}^3$ has the presentation $\langle a, b \mid a^2 = b^3 \rangle$. On the other hand, the fundamental group of any graph is of the form of free product of $\mathbb{Z}$'s. These two can not be the same. So $\mathbb{O}$ can not be homotopy equivalent to a graph and hence can not deformation retract to a graph. There is another embedding of the knotted torus where the tunnel is knotted, e.g., a solid with a knotted tunnel as shown in Figure 4a. In this case $\mathbb{I}$ does not have the same homotopy type as a graph. Figure 4b is a combination of these two cases where both $\mathbb{I}$ and $\mathbb{O}$ do not deformation retract to a graph.



(a) (b)

Figure 3: A thickened trefoil: (a) The red loop obtained by projecting the trefoil knot on the surface is not a tunnel. (b) The red loop with three windings is a tunnel.



(a) (b)

Figure 4: Surfaces that are not graph retractable : (a) a solid with a knotted tunnel; it does not have the homotopy type of any graph. (b) a knotty cup; both inside and outside are not homotopy equivalent to any graph.

We consider connected closed surfaces in $\mathbb{S}^3$ for which $\mathbb{I}$ and $\mathbb{O}$ deformation retract to graphs. We call them *graph retractable* surfaces. Since tunnel and handle loops on non graph retractable surfaces may not have natural meaning as we explained, this is not a serious restriction in practice. A large class of surfaces in practice are graph retractable.

Let $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ denote the two core graphs to which $\mathbb{I}$ and $\mathbb{O}$ deformation retract respectively. Both $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ are connected. Since $\mathbb{I}$ and $\mathbb{O}$ deformation retract to $(\mathbb{I}\setminus\Sigma)$ and $(\mathbb{O}\setminus\Sigma)$ respectively, we can assume $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ lie inside $(\mathbb{I}\setminus\Sigma)$ and $(\mathbb{O}\setminus\Sigma)$ respectively.

We state two related lemmas here which are used later in the proofs.

Lemma 1 *Let $\{\ell_j\}_{j=1}^n$ be a set of loops on a topological space X such that $\{[\ell_j]\}_{j=1}^n$ form a basis for $H_1(X)$. Let A be a path connected subspace of X and $i : A \to X$ be the inclusion map. If the induced map $i_* : H_1(A) \to H_1(X)$ is an isomorphism, there exists a set of loops $\{h_j\}_{j=1}^n$ in A such that $i_*([h_j]) = [\ell_j]$ for $1 \le j \le n$. Hence $\{[h_j]\}_{j=1}^n$ form a basis for $H_1(A)$ and if we consider each h_j as a loop in X, $\{[h_j]\}_{j=1}^n$ form a basis for $H_1(X)$ too.*

Proof 2 *Since i_* is an isomorphism, there exists an element in $H_1(A)$, say τ, such that $[\ell_j] = i_*(\tau)$ for each j, $1 \le j \le n$. Since A is path connected, there exists a loop, say h_j, such that $\tau = [h_j]$. The lemma follows.*

Lemma 2 *Let $X \subset \mathbb{S}^3$ be closed. If X deformation retracts to another closed subset $A \subset \mathbb{S}^3$, then the inclusion map $i : \mathbb{S}^3 \setminus X \to \mathbb{S}^3 \setminus A$ induces an isomorphism $i_* : H_1(\mathbb{S}^3 \setminus X) \to H_1(\mathbb{S}^3 \setminus A)$.*

Proof 3 *We have*

$$H_1(\mathbb{S}^3 \setminus X) \backsimeq H^1(X) \backsimeq H^1(A) \backsimeq H_1(\mathbb{S}^3 \setminus A)$$

where $H^1(X)$ and $H^1(A)$ are the first cohomology groups of X and A respectively. The left and the right isomorphisms follow from the Alexander duality. The middle one follows as X deformation retracts to A. In addition, all three isomorphisms are natural with respect to inclusion. The lemma follows.

5 Graph complement basis through linking

Since we assume both $\mathbb{I}$ and $\mathbb{O}$ deformation retract to core graphs, we study the first homology groups of graphs and their complements in $\mathbb{S}^3$. Consider a knot [1] $K \subset \mathbb{S}^3$. Since K is homeomorphic to a circle S^1, one can assign an orientation to it. It is known that $H_1(K) = \mathbb{Z}$ and $[K]$ is the generator for $H_1(K)$. We also know that $H_1(\mathbb{S}^3 \setminus K) = \mathbb{Z}$. Let J be another knot in $\mathbb{S}^3$ that is disjoint from K. The union $J \cup K$ is a *link*. Consider a regular projection of link $J \cup K$ to a plane. Each point at which J crosses under K locally projects to one of the crossings in Figure 5. We count $+1$ for the crossing on the left and -1 for the one on the right. The linking number, denoted by $lk(K,J)$, is defined to be the sum of these crossings counted as $+1$ or -1.

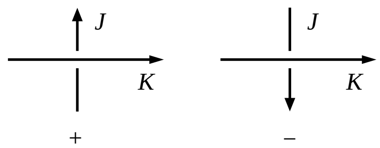


Figure 5: The two types of crossings.

Lemma 3 (Rolfsen [Rol76]) *The homology class carried by J equals nr for some generator r of $H_1(\mathbb{S}^3 \setminus K)$ if and only if $lk(K,J) = n$.*

It follows from the above lemma that if $lk(K,J) = \pm 1$, then $[J]$ is a generator for $H_1(\mathbb{S}^3 \setminus K)$.

Let G be a finite, possibly disconnected, graph in $\mathbb{S}^3$. Let T be a spanning forest of G. Let $\{e_i\}_{i=1}^n$ be the set of edges of G that are not in T. Adding any edge e_i to T form a simple loop, K_i, see Figure 6a. Each loop K_i is free of self intersections and hence can be considered as a knot. The set of knots $\{K_i\}_{i=1}^n$ may not be disjoint but generates $H_1(G)$.

Let δ_{ij} denote the Kronecker delta, that is, $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Let J_i be a knot disjoint from all K_i's and $lk(K_j, J_i) = \delta_{ij}$, see Figure 6b. It follows from Lemma 3 that $[J_i]$ is a generator of $H_1(\mathbb{S}^3 \setminus K_i)$ and is trivial in $H_1(\mathbb{S}^3 \setminus K_j)$ for $j \neq i$.

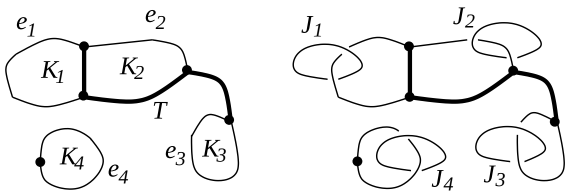


Figure 6: A graph with two components. The bold edges and the black nodes form the spanning forest.

Theorem 2 $\{[J_i]\}_{i=1}^n$ *form a basis for $H_1(\mathbb{S}^3 \setminus G)$.*

Proof 4 *Let $L_i = K_i \cup T$. By Lemma 2, the inclusion map from $\mathbb{S}^3 \setminus L_i$ to $\mathbb{S}^3 \setminus K_i$ induces an isomorphism between $H_1(\mathbb{S}^3 \setminus L_i)$ and $H_1(\mathbb{S}^3 \setminus K_i)$. Since J_i avoids G, $[J_i]$ is the generator for $\mathbb{S}^3 \setminus L_i$ but is trivial in $\mathbb{S}^3 \setminus L_j$ for any $j \neq i$.*

[1] We only consider tame knots

We prove the theorem by induction on n. Let $G_1 = \cup_{i=1}^{n-1} L_i$. Consider G and T as union and intersection of G_1 and L_n respectively. Then,

$$\begin{aligned} \mathbb{S}^3 \setminus G &= (\mathbb{S}^3 \setminus G_1) \cap (\mathbb{S}^3 \setminus L_n) \text{ and} \\ \mathbb{S}^3 \setminus T &= (\mathbb{S}^3 \setminus G_1) \cup (\mathbb{S}^3 \setminus L_n). \end{aligned}$$

By induction hypothesis, $\{[J_i]\}_{i=1}^{n-1}$ form a basis for $H_1(\mathbb{S}^3 \setminus G_1)$ and are all trivial in $H_1(\mathbb{S}^3 \setminus L_n)$. Clearly, $[J_n]$ is the generator for $H_1(\mathbb{S}^3 \setminus L_n)$ and is trivial in $H_1(\mathbb{S}^3 \setminus G_1)$. Hence $\{[J_i]\}_{i=1}^{n}$ form a basis for $H_1(\mathbb{S}^3 \setminus G_1) \oplus H_1(\mathbb{S}^3 \setminus L_n)$. Consider the following Mayer-Vietoris sequence.

$$\cdots \to H_2(\mathbb{S}^3 \setminus T) \xrightarrow{\alpha} H_1(\mathbb{S}^3 \setminus G)$$
$$\xrightarrow{\beta} H_1(\mathbb{S}^3 \setminus G_1) \oplus H_1(\mathbb{S}^3 \setminus L_n) \xrightarrow{\gamma} H_1(\mathbb{S}^3 \setminus T) \to \cdots$$

Since $H_1(\mathbb{S}^3 \setminus T) = 0$ and $H_2(\mathbb{S}^3 \setminus T) = 0$, β is an isomorphism, which is induced by the inclusion map. The theorem follows as $\{[J_i]\}_{i=1}^{n}$ form a basis for $H_1(\mathbb{S}^3 \setminus G_1) \oplus H_1(\mathbb{S}^3 \setminus L_n)$.

The proof of Theorem 2 implies that $H_1(\mathbb{S}^3 \setminus G) \overset{i_*}{\simeq} \oplus_{i=1}^{n} H_1(\mathbb{S}^3 \setminus K_i)$, which with Lemma 3 leads to the following corollary.

Corollary 4 *Let ℓ be a loop in $\mathbb{S}^3 \setminus G$. One has $[\ell] = \sum_{i=1}^{n} a_i [J_i]$ if and only if $lk(\ell, K_i) = a_i$ for each i where a_i is an integer.*

6 Minimally linked loops

In this section, we give more constructive statements about the handle and tunnel loops for a graph retractable surface. These statements relate the linking numbers of the loops with the core graphs to their characterization and existence. Theorem 3 characterizes the handle and tunnel loops in terms of linking numbers. Theorem 4 provides the existence of a special class of handle and tunnel loops which link minimally with the core graphs. This leads to an algorithm to compute a set of $2g$ loops forming a minimally linked basis for $H_1(M)$ where g of them are handle and the other g are tunnel.

Consider the union of two core graphs $\widehat{\mathbb{I}} \cup \widehat{\mathbb{O}}$ as a single graph with two connected components. From the previous discussion, we can compute a set of knots, denoted $\{K_i\}_{i=1}^{2g}$, which generate $H_1(\widehat{\mathbb{I}} \cup \widehat{\mathbb{O}})$. Let $\{K_j^{\mathbb{I}}\}_{j=1}^{g}$ denote the loops from $\widehat{\mathbb{I}}$ and $\{K_j^{\mathbb{O}}\}_{j=1}^{g}$ denote the loops from $\widehat{\mathbb{O}}$.

Theorem 3 *A loop ℓ on a graph retractable surface M is a handle if and only if $lk(\ell, K_i^{\mathbb{I}}) \neq 0$ for at least one of $K_i^{\mathbb{I}}$'s and $lk(\ell, K_i^{\mathbb{O}}) = 0$ for all the $K_i^{\mathbb{O}}$'s. A loop ℓ on M is a tunnel if and only if $lk(\ell, K_i^{\mathbb{O}}) \neq 0$ for at least one of $K_i^{\mathbb{O}}$'s and $lk(\ell, K_i^{\mathbb{I}}) = 0$ for all the $K_i^{\mathbb{I}}$'s.*

Proof 5 *Recall that Σ denotes a tubular neighborhood of M in $\mathbb{S}^3$. We have*

$$H_1(\mathbb{O}) \overset{i_*}{\simeq} H_1(\mathbb{S}^3 \setminus (\mathbb{I} \setminus \Sigma)) \overset{i_*}{\simeq} H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{I}}). \tag{3}$$

The first isomorphism holds since $\mathbb{S}^3 \setminus (\mathbb{I} \setminus \Sigma)$ deformation retracts to $\mathbb{O}$. The second isomorphism follows from Lemma 2 as $(\mathbb{I} \setminus \Sigma)$ deformation retracts to $\widehat{\mathbb{I}}$. In addition, both isomorphisms are induced by inclusion maps. Symmetrically we have

$$H_1(\mathbb{I}) \overset{i_*}{\simeq} H_1(\mathbb{S}^3 \setminus (\mathbb{O} \setminus \Sigma)) \overset{i_*}{\simeq} H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{O}}). \tag{4}$$

By definition, a loop ℓ on M is a handle one if and only if $[\ell]$ is trivial in $H_1(\mathbb{I})$ and non trivial in $H_1(\mathbb{O})$, which means, by Equation 3 and 4, it is trivial in $H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{O}})$ and non trivial in $H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{I}})$. From Corollary 4, the homology class of ℓ in $H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{O}})$ is trivial if and only if $lk(\ell, K_i^{\mathbb{O}}) = 0$ for each $1 \le i \le g$, and the homology class of ℓ in $\mathbb{S}^3 \setminus \widehat{\mathbb{I}}$ is non trivial if and only if at least one of $lk(\ell, K_i^{\mathbb{I}})$ is non zero. A similar argument applies to tunnel loops.

Next, we prove the existence of a set of $2g$ loops on M which link the core graphs in a minimal sense. We will prove that these loops generate all the handles and tunnels in theorem 4.

Definition 3 *A loop ℓ on surface M is* minimally linked *if $\sum_{j=1}^{2g} |lk(\ell, K_j)| = 1$.*

Intuitively, minimal linking forbids the loops to wind around the surface multiple times thereby producing more meaningful handle and tunnel loops.

Lemma 5 *There exists a minimally linked set of loops $\{J_i\}_{i=1}^{2g}$ on a graph retractable surface M where $\{[J_i]\}_{i=1}^{2g}$ form a basis for $H_1(M)$.*

Proof 6 *It is obvious that there exists a set of loops in $\mathbb{S}^3 \setminus (\widehat{\mathbb{I}} \cup \widehat{\mathbb{O}})$, denoted $\{\ell_i\}_{i=1}^{2g}$, such that $|lk(\ell_i, K_j)| = \delta_{ij}$ for any $1 \le i, j \le 2g$. For example, ℓ_i can be chosen arbitrarily close to K_i to loop around it once and avoid linking any other K_j, $j \ne i$. From Theorem 2, $\{[\ell_i]\}_{i=1}^{n}$ form a basis for $H_1(\mathbb{S}^3 \setminus (\widehat{\mathbb{I}} \cup \widehat{\mathbb{O}}))$. We have*

$$H_1(M) \overset{i_*}{\simeq} H_1(\Sigma) \overset{i_*}{\simeq} H_1(\mathbb{S}^3 \setminus (\widehat{\mathbb{I}} \cup \widehat{\mathbb{O}})). \tag{5}$$

The first isomorphism holds since Σ is a tubular neighborhood of M. The second isomorphism follows from Lemma 2 as $(\mathbb{I} \setminus \Sigma) \cup (\mathbb{O} \setminus \Sigma)$ deformation retracts to $\widehat{\mathbb{I}} \cup \widehat{\mathbb{O}}$. In addition both isomorphisms are induced by inclusion maps. From Lemma 1, there exists a loop J_i on M for $1 \le i \le n$ such that $[J_i] = [\ell_i]$. Hence we have $|lk(J_i, K_j)| = |lk(\ell_i, K_j)| = \delta_{ij}$. Furthermore $\{[J_i]\}_{i=1}^{n}$ form a basis for $H_1(M)$.

We observe that the above lemma holds even if we require J_i to pass through a specific point, say p. This is because we assume surface M is connected and hence path connected. Let J_i be a loop guaranteed by Lemma 5. Let h be the path from p to a point on J_i and $\overline{h}$ be the reverse of h. Then the loop ℓ formed by concatenating h, J_i and $\overline{h}$ can be taken as new J_i that passes through p. Furthermore, when M is a polygonal mesh, one can restrict each J_i to the edges of M. This is because we can perform a finite sequence of deformation each of which is within a polygon to deform a J_i in a general position to the one along the edges without changing its homology class.

The set of loops $\{J_j\}_{j=1}^{2g}$ defined in Lemma 5 can be classified into two groups: $\{J_j^{\mathbb{I}}\}_{j=1}^{g}$ with $|lk(J_j^{\mathbb{I}}, K_j^{\mathbb{I}})| = 1$ for $1 \le j \le g$, and $\{J_j^{\mathbb{O}}\}_{j=1}^{g}$ with $|lk(J_j^{\mathbb{O}}, K_j^{\mathbb{O}})| = 1$ for $1 \le j \le g$. The homology classes $\{[K_j^{\mathbb{I}}]\}_{j=1}^{g}$, $\{[J_j^{\mathbb{I}}]\}_{j=1}^{g}$, $\{[K_j^{\mathbb{O}}]\}_{j=1}^{g}$ and $\{[J_j^{\mathbb{O}}]\}_{j=1}^{g}$ form a basis for $H_1(\widehat{\mathbb{I}})$, $H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{I}})$, $H_1(\widehat{\mathbb{O}})$ and $H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{O}})$, respectively. Furthermore, due to Corollary 4 each $[J_j^{\mathbb{I}}]$ is trivial in $H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{O}})$ and so is each $[J_j^{\mathbb{O}}]$ in $H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{I}})$. Next theorem formally establishes that there exist minimally linked handle and tunnel loops forming a basis for $\mathbb{I}$ and $\mathbb{O}$ respectively.

Theorem 4 *There exist a set of minimally linked handle loops on a graph retractable surface which form a basis for $H_1(\mathbb{O})$. Similarly, there is a set of minimally linked tunnel loops which form a basis for $H_1(\mathbb{I})$.*

Proof 7 *Consider the loops $\{[J_j^{\mathbb{I}}]\}_{j=1}^{g}$ and $\{J_j^{\mathbb{O}}\}_{j=1}^{g}$. By Equation 3, $\{[J_j^{\mathbb{I}}]\}_{j=1}^{g}$ form a basis for $H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{I}})$ and hence for $H_1(\mathbb{O})$ while $\{[J_j^{\mathbb{O}}]\}_{j=1}^{g}$ are trivial in $H_1(\mathbb{S}^3 \setminus \widehat{\mathbb{I}})$ and hence are trivial in $H_1(\mathbb{O})$. Symmetrically by Equation 4, we have that $\{[J_j^{\mathbb{I}}]\}_{j=1}^{g}$ are trivial in $H_1(\mathbb{I})$ while $\{[J_j^{\mathbb{O}}]\}_{j=1}^{g}$ form a basis for $H_1(\mathbb{I})$. The theorem follows from the definitions of tunnel loop and handle loop.*

7 Topological algorithms

In this section, we present algorithms to *detect* and *generate* handle and tunnel loops for a graph retractable surface.

Assume $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ are given. In a preprocessing step we compute a set of knots $\{K_i\}_{i=1}^{2g}$ from $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ using their spanning trees as indicated in section 5. Assuming that $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ have t edges altogether, computation of K_is take $O(t)$ time.

DETECTION: Let ℓ be any given loop with s edges on M. By Theorem 3, if ℓ links $\widehat{\mathbb{I}}$ and not $\widehat{\mathbb{O}}$, it is a handle loop. If ℓ links $\widehat{\mathbb{O}}$ and not $\widehat{\mathbb{I}}$, it is a tunnel loop. If none of these conditions holds, ℓ is neither a handle nor a tunnel. The time complexity of this detection is dominated by the linking number computation. The linking numbers can be determined by computing the intersections between edges of ℓ and K_is projected on a plane. It can be done in time $O(t+s+x)\log(t+s+x)$ where x is the number of total intersections.

GENERATION: Here we compute a set of minimally linked g handle and tunnel loops whose homology classes form a basis for $H_1(\mathbb{O})$ and $H_1(\mathbb{I})$ respectively. Because of minimal linking property, the handle and tunnel loops cannot wind around the surface arbitrarily, a property that goes with the intuitive meaning of handles and tunnels. Assume that M has n edges. We use any of the $O(n)$ algorithms [DS95, VY90] to compute a system of $2g$ loops on M, $\{\ell_i\}_{i=1}^{2g}$, through a fixed point p. The homology classes carried by these loops form a basis for $H_1(M)$. For each loop ℓ_i, we compute the linking number $lk(\ell_i, K_j)$ for each K_j. Let $\{J_j\}_{j=1}^{2g}$ be the loops guaranteed by Theorem 4. It follows from Corollary 4 that

$$[\ell_i] = \sum_{j=1}^{2g} lk(\ell_i, K_j)[J_j].$$

Let A be the matrix $\{lk(\ell_i, K_j)\}$, $\vec{\mathscr{L}}$ be the vector $[\ [\ell_1], \cdots, [\ell_{2g}]\ ]^t$ and $\vec{\mathscr{J}}$ be the vector $[\ [J_1], \cdots, [J_{2g}]\ ]^t$. We have

$$\vec{\mathscr{L}} = A\vec{\mathscr{J}}.$$

Since both $\{[\ell_i]\}_{i=1}^{2g}$ and $\{[J_i]\}_{i=1}^{2g}$ are bases for $H_1(M)$, A is invertible. Let $A^{-1} = \{a_{ij}\}$. For $1 \le i \le 2g$, J_i can be uniquely expressed as

$$[J_i] = \sum_{j=1}^{2g} a_{ij}[\ell_j].$$

Observe that $[J_i]$ is some linear combination of $[\ell_j]$'s with integer coefficients. So the entries in A^{-1} must be integers.

Since all ℓ_j's have the same base point, we can concatenate any two of them, say ℓ_i and ℓ_j, into a new loop, denoted $\ell_i + \ell_j$. For an integer z, let $z\ell_i$ denote the loop formed by concatenating $|z|$ copies of ℓ_i if $z > 0$ or $|z|$ copies of the reverse of ℓ_i if $z < 0$. Then we can take J_i as $\sum_{j=1}^{2g} a_{ij}\ell_j$ for each $1 \le i \le 2g$. From Theorem 3, we know J_i is a handle loop if K_i is in $\widehat{\mathbb{I}}$ or is a tunnel loop if K_i is in $\widehat{\mathbb{O}}$.

The time complexity of the generation algorithm is dominated by the linking number computation and the matrix inversion which take $O((gn+t+x)\log(gn+t+x)+g^3)$ time since each ℓ_i has $O(n)$ edges.

8 Incorporating geometry

In this section, we give an implementation of the algorithm assuming that M is presented as a piecewise linear surface. The algorithm described in section 7 computes topologically correct handle and tunnel loops without considering any geometric information. Hence the computed loops may not be very good geometrically. The implementation presented in this section incorporates geometry into the algorithm, namely it computes two sets of loops with small size. Although it does not guarantee that the computed set of handle or tunnel loops is the shortest, the experiments show that they are geometrically meaningful.

8.1 Computing core graphs and $\{K_i\}_{i=1}^{2g}$

We use Voronoi diagrams for computing $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$. Let V be the set of vertices in M and Vor$V$ denote the Voronoi diagram of V. Let $\mathbb{I}$ and $\mathbb{O}$ denote the bounded and unbounded spaces respectively bordered by M as defined earlier. Consider the 2-complex $\mu_{\mathbb{I}}$ formed by the Voronoi facets and its lower dimensional faces that lie completely in $\mathbb{I}$. Similarly define $\mu_{\mathbb{O}}$ corresponding to $\mathbb{O}$. It is known that if V samples M densely, $\mu_{\mathbb{I}}$ and $\mu_{\mathbb{O}}$ are homotopy equivalent to $\mathbb{I}$ and $\mathbb{O}$ respectively [CL04, GRS06].

The basic idea to compute $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ is to collapse $\mu_{\mathbb{I}}$ and $\mu_{\mathbb{O}}$ respectively from their boundaries. A basic collapse operation deletes a pair of faces preserving a homotopy equivalence between the spaces before and after the collapse. At the end of all possible collapses, we obtain two graphs that are homotopy equivalent to $\mathbb{I}$ and $\mathbb{O}$. These two graphs can be taken as $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$.

Although any collapse order is permissible, we use a specific one proposed in Dey and Sun [DS06] for computing the so called curve-skeletons. These curve-skeletons are graphs that reside roughly in the 'middle' of $\mathbb{I}$ and $\mathbb{O}$ thereby capturing shape geometry better. A real valued function called *medial geodesic* defined on the faces of $\mu_{\mathbb{I}}$ and $\mu_{\mathbb{O}}$ guides the collapse. For each Voronoi facet f, the value is the shortest geodesic distance on M between the end points of the Delaunay edge dual to f. The Voronoi edges and vertices get a value which is maximal among all values of the Voronoi facets incident to them. After collapsing $\mu_{\mathbb{I}}$ and $\mu_{\mathbb{O}}$ ordered with increasing medial geodesic values, one obtains the edges of the curve-skeletons. Each such edge gets associated with an additional value called geodesic size which indicates the local size of M. We use the curve-skeletons and this additional value to compute handle and tunnel loops of small size.

In our implementation, the geodesic distance between two points on M is approximated by the Dijkstra's shortest path over the graph consisting of the edges of M. Each skeleton edge on the resulting curve skeletons is a Voronoi edge whose dual Delaunay triangle has three vertices on surface M. The geodesic paths between each pair of them together form a 'circle', called the geodesic circle. Its length is the geodesic size associated with the corresponding skeleton edge, which indicates how big the shape is locally.

We obtain $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ by imposing a graph structure on both curve-skeletons, namely, taking the vertices of degree more than two as the graph nodes and forming a graph edge by a sequence of connected skeleton edges in between two such nodes.

There are surfaces for which one can not compute graphs $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ by collapsing $\mu_{\mathbb{I}}$ and $\mu_{\mathbb{O}}$ respectively even when both $\mathbb{I}$ and $\mathbb{O}$ deformation retract to graphs. As mentioned in [DS06], one such example can be derived from the famous "house with two rooms" [Hat02]. Computing $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ for such pathological cases remains an open question.

Define the geodesic size for a graph edge $E \in \widehat{\mathbb{I}} \cup \widehat{\mathbb{O}}$ as the smallest geodesic size among the skeleton edges contained in that graph edge, i.e.,

$$g(E) = min\{g(e) : \text{e is a skeleton edge in E}\}$$

where $g(e)$ is the geodesic size associated with the skeleton edge e. We obtain maximal spanning trees for $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$, which form a spanning forest for $\widehat{\mathbb{I}} \cup \widehat{\mathbb{O}}$. We have g edges left in $\widehat{\mathbb{I}}$ and g edges left in $\widehat{\mathbb{O}}$. Let $\{E_i\}_{i=1}^{2g}$ denote these edges. As discussed in Section 5, adding E_i, $i = 1, \ldots, 2g$, back to the spanning forest form a set of knots K_i, $i = 1, \ldots, 2g$.

8.2 Computing handle and tunnel loops $\{J_i\}_{i=1}^{2g}$

All J_i's computed by the algorithm in Section 7 pass through an arbitrarily chosen common point. Hence their sizes are usually not small. Of course, one can use the algorithm of Lazarus and Verdière [ECdVL05] to compute a set of shortest loops with the same homotopy type of the loops computed by the algorithm in Section 7. However, the restriction that they have to pass through a single base point make them usually long. In our implementation, we compute each J_i at different places corresponding to the skeleton edges with small geodesic size. As a result these loops tend to be small. To compute J_i, we consider E_i. Let e be the skeleton edge in E_i having the smallest geodesic size. The edge e is a Voronoi edge. Let p be one of the vertices of the dual Delaunay triangle of e. Use the method proposed by Erickson and Whittlesey [EW05] to compute a system of loops through p and follow the algorithm stated in Section 7 to obtain J_i. Hence for each i, $1 \leq i \leq 2g$, we compute a J_i with $lk(J_i, K_j) = \delta_{ij}$. By Theorem 4, we have computed a set of g handle loops and g tunnel loops.

Erickson and Whittlesey present two methods to obtain a system of loops through a fixed point. One of these methods computes these $2g$ loops one by one in the following way. It runs the Dijkstra's shortest-path algorithm with starting point p. Whenever the wave front of equi-distant vertices touches itself as it sweeps across the surface, a loop is formed by the two paths from p to the touching point. The algorithm checks whether this loop is contractible. If not, it continues to propagate the wave front until a non contractible loop is found. This loop, denoted ℓ_1, is actually the shortest non contractible loop through p. The surface is cut along ℓ_1 and then the algorithm computes another non contractible loop, denoted ℓ_2, from one copy of p to the other on $M \setminus \ell_1$. This process of finding and cutting is repeated to find $2g$ loops, which form a system of loops through p. In our implementation, we compute $lk(\ell_j, K_i)$ for $1 \leq i \leq 2g$ immediately after we obtain ℓ_j. In most practical cases, some ℓ_j itself satisfies the condition to be J_i, namely $lk(\ell_j, K_i) = \pm 1$ and $lk(\ell_j, K_m) = 0$ with $m \neq i$. In such case, we do not need to continue to compute the remaining ℓ_j's.

The way we choose the starting point p makes sure that the computed ℓ_j's are of small size. Figure 7 shows the tunnel loops and the handle loops together with $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$ computed by the above implementation. Notice that the models shown in the second row are both double torus. Notice how the difference in their embedding makes a difference in the detected tunnel loops. As one can observe, both tunnel loops for DoubleTorus 2 manage to link only with one loop from $\widehat{\mathbb{O}}$.

Figure 8 and Figure 10 show only the tunnel and the handle loops for some more complicated models. As we can see, the reconstructed Buddha model contains three extra small holes.

9 Application

In this section we apply handle and tunnel loops to compute actual handle or tunnel features for shapes which can further be used for removing insignificant topologies.

Feature detection. The basic idea is to sweep the handle or tunnel loops over the surface appropriately. To compute a tunnel feature, we start with a tunnel loop. We run Dijkstra's shortest path algorithm for multiple sources where the starting points are the vertices on the tunnel loop. At any generic step Dijkstra's algorithm maintains a wave front of shortest distant on each side of the tunnel loop. We keep propagating these two wave fronts until one of the following conditions does not hold.

- The wave front links the core graph exactly the same way as the initial tunnel loop does, i.e., if ℓ and t denote the wave front and the tunnel respectively, then $lk(\ell, K_i) = lk(t, K_i)$ for each K_i.
- The ratio between the length of the wave front and that of the initial tunnel loop does not exceed a given threshold.

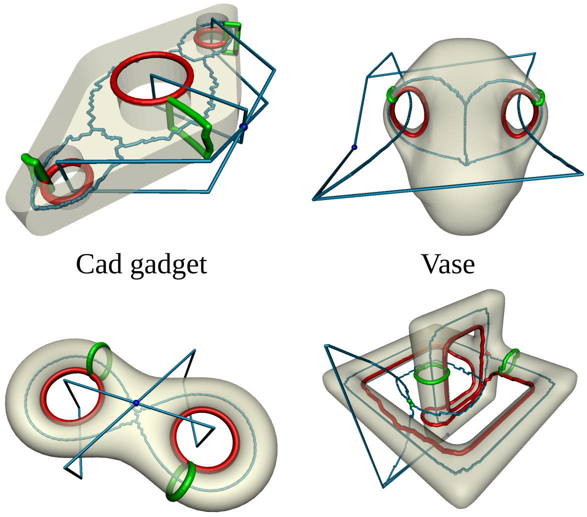


Figure 7: The handle loops (green) and the tunnel loops (red) together with $\widehat{\mathbb{I}}$ and $\widehat{\mathbb{O}}$. Those edges in $\widehat{\mathbb{O}}$ connected to the infinite point are routed to a point outside the bounding boxes of the objects.

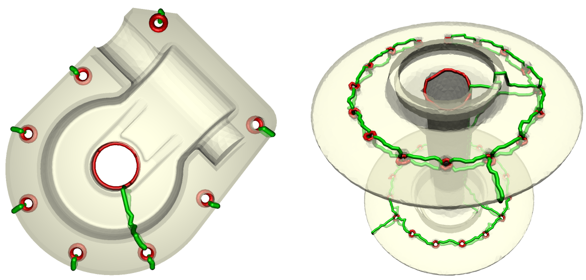


Figure 8: The handle loops (green) and the tunnel loops (red).

We take the region swept by these two fronts as the tunnel feature corresponding to the initial tunnel loop. Similarly one can compute a handle feature by starting from a handle loop. Figure 9 shows the resulting tunnel and handle features. The ratio threshold is set to be 1.2 for all the examples.

Topology simplification. We observe that our method can be extended further to remove (insignificant) topologies from a shape [BK05, ESV97, GW01, WHDS04, ZJH07]. We first identify handle or tunnel features that need to be removed. For example, one may consider the lengths of the handle and tunnel loops to determine which features to be removed. We assume that the surface is a Delaunay subcomplex made out of Delaunay triangles. Many algorithms in surface reconstruction and mesh generation produce such surfaces. Even non Delaunay surfaces can be converted to a Delaunay one by resampling and remeshing techniques. The Delaunay tetrahendra inside the shape represents its volume. To fill a small tunnel, we add those tetrahendra back to the volume that have all four vertices on the corresponding tunnel feature. Figure 9 and Figure 10 show that holes get filled by this method for Casting, Fusee and Buddha. Similarly we can cut a handle by deleting those tetrahendra with all four vertices on that handle feature from the volume representation. Figure 9 shows such a removal of handles from Vase.

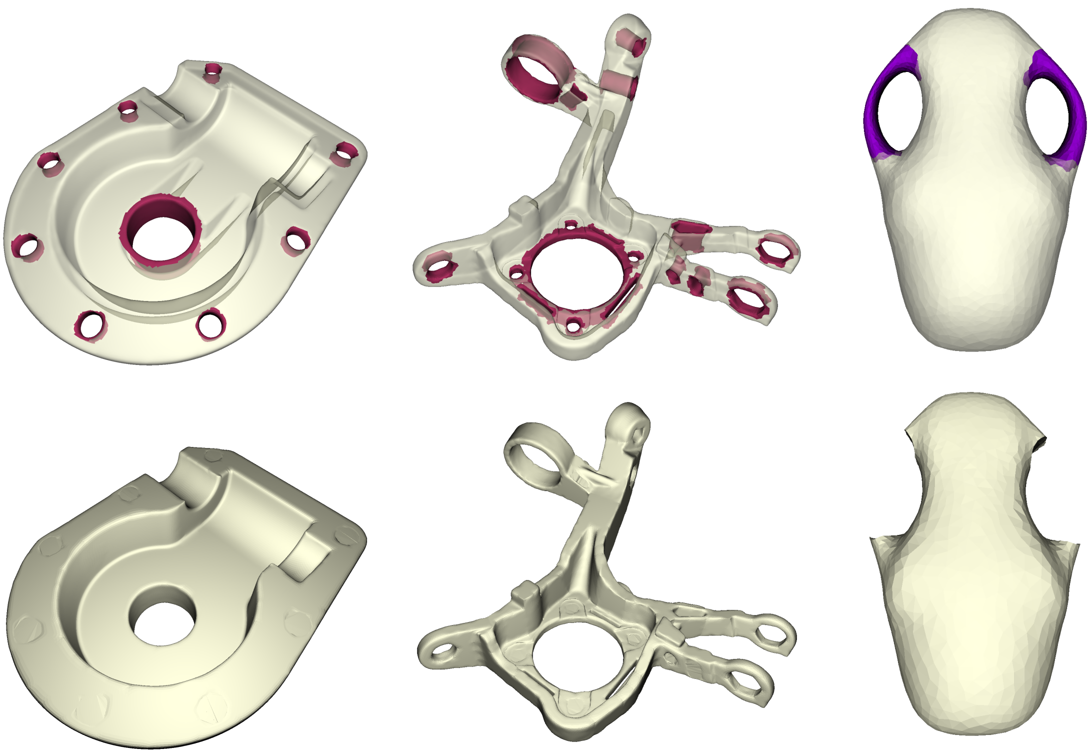


Figure 9: Top row: the tunnel features for Casting and Fusee and the handle features for Vase. Bottom row: small tunnels are filled for Casting and Fusee ; both handles in Vase are removed.

10 Conclusions and future work

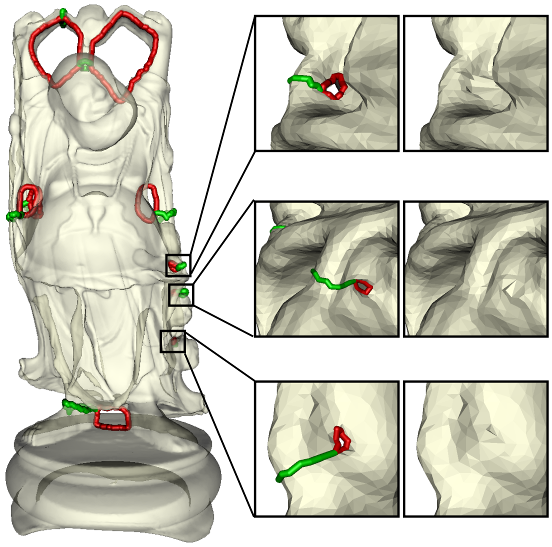
In this work we define and prove the existence of loops on surfaces which represent handles and tunnels for the shape bordered by the surface. We characterize these loops on graph retractable surfaces using linking with core graphs. These characterizations lead to algorithms for detecting and generating handle and tunnel loops.

Several open questions arise as a result of this research. Our implementation does not guarantee that the computed tunnel or handle loops are the shortest in length. Computing a set of shortest handle and tunnel loops among all possible such loops remains an open challenge. Another natural question is to compute handle and tunnel loops on non graph retractable surfaces. Our definition of handle and tunnel loops depend on the fact that the surface has no boundary. Is it possible to extend the ideas of this paper to surfaces with boundaries?

References

[BK05] BISCHOFF S., KOBBELT L.: Structure preserving cad model repair. *Comput. Graphics Forum 24* (2005), 527–536.

[CL04] CHAZAL F., LIEUTIER A.: Stability and homotopy of a subset of the medial axis. In *Proc. ACM Sympos. Solid Modeling and Applications* (2004), pp. 243–248.



Buddha (g = 9)

Figure 10: Three closeup views of small tunnels in Buddha which are filled up.

[DS95] DEY T. K., SCHIPPER H.: A new technique to compute polygonal schema for 2-manifolds with application to null-homotopy detection. *Discrete Comput. Geom. 14* (1995), 93–110.

[DS06] DEY T. K., SUN J.: Defining and computing curve-skeletons with medial geodesic function. In *Proc. Sympos. Geom. Processing* (2006), pp. 143–152.

[ECdVL05] ÉRIC COLIN DE VERDIÈRE, LAZARUS F.: Optimal system of loops on an orientable surface. *Discrete Comput. Geom. 33* (2005), 507–534.

[EHP04] ERICKSON J., HAR-PELED S.: Optimally cutting a surface into a disk. *Discrete Comput. Geom. 31* (2004), 37–59.

[ESV97] EL-SANA J., VARSHNEY A.: Controlled simplification of genus for polygonal models. In *Proc. IEEE Visualization'97* (1997), pp. 403–412.

[EW05] ERICKSON J., WHITTLESEY K.: Greedy optimal homotopy and homology generators. In *Proc. ACM-SIAM Sympos. Discrete Algorithms* (2005), pp. 1038–1046.

[GRS06] GIESEN J., RAMOS E. A., SADRI B.: Medial axis approximation and unstable manifolds. In *Proc. 22nd Annu. Sympos. Comput. Geom.* (2006), pp. 327–336.

[GW01] GUSKOV I., WOOD Z.: Topological noise removal. In *Proc. Graphics Interface 2001* (2001), pp. 19–26.

[Hat02] HATCHER A.: *Algebraic Topology*. Cambridge University Press, 2002.

[Moi77] MOISE E. E.: *Geometric Topology in Dimensions 2 and 3*. Springer-Verlag, 1977.

[Mun93] MUNKRES J. R.: *Elements of Algebraic Topology*. Addison Wesley, 1993.

[Rol76] ROLFSEN D.: *Knots and Links*. Publish or Perish, 1976.

[VY90] VEGTER G., YAP C. K.: Computational complexity of combinatorial surfaces. In *Proc. Annu. Sympos. Comput. Geom.* (1990), pp. 102–111.

[WHDS04] WOOD Z., HOPPE H., DESBRUN M., SCHRÓDER P.: Removing excess topology from isosurfaces. *ACM Trans. Graphics 23* (2004), 511–533.

[ZJH07] ZHOU Q.-Y., JU T., HU S.-M.: Topology repair of solid models using skeletons. *IEEE Transactions on Visualization and Computer Graphics* (2007).