Stability of Topological Persistence for Domains *

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Abstract

Scalar functions defined on a topological space $\Omega$ are at the core of many applications such as shape matching, visualization and physical simulations. Topological persistence is an approach to characterizing these functions. It measures how long topological structures in the level sets $\{x \in \Omega : f(x) = c\}$ persist as $c$ changes. Recently it was shown that the critical values defining a topological structure with relatively large persistence remain almost unaffected by small perturbations. This result suggests that topological persistence is a good measure for matching and comparing scalar functions. We extend these results to critical points in the domain by redefining persistence and critical points and replacing sub-level sets $\{x \in \Omega : f(x) \leq c\}$ with interval sets $\{x \in \Omega : a \leq f(x) < b\}$. With these modifications we establish a stability result for domain points that can be used for matching two scalar functions.

1 Introduction

A scalar field is a scalar function $f : \Omega \to \mathbb{R}$ defined on some topological space $\Omega$. Examples of scalar fields are fluid pressure in computational fluid dynamics simulations, temperature in oceanographic or atmospheric studies, and density in medical CT or NMR scans. A level set of a scalar field is a set of points with the same scalar value, i.e., $\{x \in \Omega : f(x) = c\}$. One way of deriving quantitative information about scalar fields is by studying the topological structures of its level sets or the regions bounded by level sets, such as $\{x \in \Omega : f(x) \leq c\}$. The mathematical field of Morse Theory is the study of these topological structures.

Among the most basic problems on scalar fields is simplifying a scalar field for compact representation, identifying important features in a scalar field, and characterizing the essential structure of a scalar field. Extracting and representing the topological structure of the level sets is one way of approaching all these problems. However, this topological structure may contain “small” topological features which are insignificant or caused by noise. Small topological features should be removed in simplification and ignored in characterizing essential structure or identifying important features. How does one determine which topological features are small?

Edelsbrunner, Letscher and Zomorodian in [5] introduced the notion of topological persistence. As $c \in \mathbb{R}$ increases, topological features appear and disappear in the set $\{x \in \Omega : f(x) \leq c\}$. If a topological feature appears at “time” $a$ and disappears at “time” $b$, then its persistence is the difference, $b - a$. Between these two times, Edelsbrunner et al. in [5] use homology groups over $\mathbb{Z}/2\mathbb{Z}$ to define topological features. Carlsson and Zomorodian [9] showed how topological persistence could be computed for homology groups over any fields.

At the core of various application areas such as shape matching and visualization is the problem of characterizing and comparing scalar fields. Topological persistence gives one approach to comparing such fields. Two fields are similar if they have matching topological features with approximately the same persistence. This approach to comparing fields makes sense only if persistence remains stable under relatively small perturbations of the scalar fields. Cohen-Steiner, Edelsbrunner and Harer [3] proved that “large” persistence values remain almost unaffected. More precisely, let scalar field $\tilde{f} : \Omega \to \mathbb{R}$ be a small perturbation of field $f : \Omega \to \mathbb{R}$, (i.e., $|f(x) - \tilde{f}(x)| \leq \delta$ for all $x \in \Omega$.) If $f$ has a topological structure with relatively large persistence which appears at $a$ and disappears at $b$, then $\tilde{f}$ has a corresponding topological structure which appears around $a$ and disappears around $b$.

Critical values are the range scalar values where the topological structure of the level sets changes. Cohen-Steiner et. al. showed that the critical values for structures with large persistence remain stable under relatively small perturbations of the scalar field. Scalar fields also have critical points, points in the domain which change the topological structure of the level sets. It is natural to ask if critical points for structures with large persistence remain stable under perturbations of the field. If two scalar fields are close, then are their significant critical points “close”?

In this paper we revisit topological persistence and establish a stability result in terms of the critical points in

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the domain. There are two obstacles that we must overcome. First, if we look only at the topological structure of \( \{ x \in \Omega : f(x) \leq c \} \), then there is no such stability result.

Consider Figure 1. Let \( f \) be a function defined on the surface in \( \mathbb{R}^3 \) shown in the figure where \( f(x) \) is the \( z \)-coordinate of the point \( x \). The set \( \{ x \in \Omega : f(x) \leq f(r) \} \) is homeomorphic to a cylinder with circles \( c_1 \) and \( c_2 \) bounding each end. The first homology group (\( H_1 \)) is generated by the circle \( c_1 \). (It is also generated by circle \( c_2 \).) Assume that the two maxima \( p \) and \( q \) have \( z \)-coordinates that are almost equal except that \( f(p) < f(q) \).

Since \( f(p) < f(q) \), the first homology group becomes trivial in the set \( \{ x \in \Omega : f(x) \leq f(p) \} \). Loosely speaking, the cycle generated by \( c_1 \) at \( r \) gets destroyed at \( q' \) in \( \{ x \in \Omega : f(x) \leq f(q') \} \). We get a persistent value pair \([f(r), f(p)]\).

Now consider a slightly perturbed \( f \) denoted as \( \hat{f} \). Set \( \hat{f} \) equal to \( f \) everywhere except in the vicinity of \( p \) and \( q \) where \( \hat{f} \) is perturbed so that \( f(q) < f(p) \). Let \( \hat{p} \) and \( \hat{q}' \) be new maxima close to \( p \) and \( q \) respectively for \( \hat{f} \). The cycle generated by \( c_1 \) at \( r \) gets destroyed at \( \hat{q}' \) in \( \{ x \in \Omega : f(x) \leq f(q') \} \). We get a persistent value pair \([f(r), f(q')]\) for \( \hat{f} \). Since \( f(p) \) is close to \( f(q') \), the two persistent value pairs \([f(r), f(p)]\) and \([\hat{f}(r), \hat{f}(q')]\) are close, confirming the Cohen-Steiner et al. result. However, the points \( p \) and \( q' \) are not close in any sense in the domain.

Instead of considering only sets \( \{ x \in \Omega : f(x) \leq c \} \) which are bounded from above by a single level set, we will consider interval sets that are bounded from above and below by level sets. The sets we use are \( \{ x \in \Omega : a \leq f(x) < b \} \). This is one crucial deviation we make from the set up in earlier works [3, 5, 9]. It also leads to slightly different definitions for critical points and persistence.

Returning to Figure 1, the first homology group of the set \( \{ x : f(r) \leq f(x) < f(p) \} \) (open at the top) has two distinct generators, one given by circle \( c_1 \) and one by circle \( c_2 \). Point \( p \) destroys the homology group generated by \( c_1 \) while point \( q \) destroys the one generated by \( c_2 \). In the perturbed field \( \hat{f} \) where \( f(q') < f(q') \), a similar thing happens. Thus \( p \) and \( q' \) (also \( q \) and \( q' \)) are critical points for \( f \) and \( \hat{f} \) respectively and destroy homology groups with approximately the same persistence.

The second problem in stability of critical points is that corresponding critical points can be arbitrarily far apart. Consider functions \( f \) and \( \hat{f} \) in Figure 2 where \( |f(x) - \hat{f}(x)| < \delta \) for all \( x \in \Omega \). The maxima, \( p \) and \( q' \), of \( f \) and \( \hat{f} \), respectively, can be made arbitrarily far apart even as \( \delta \) is made arbitrarily small.

\[ \text{Figure 2: The maximum } p \text{ for a real valued function } f \text{ has moved by large distance even for an arbitrarily close approximant } \hat{f}. \]

Instead of using a metric in the domain, we use the range to determine neighborhoods of points. A neighborhood of point \( p \) is the connected component of \( \{ x \in \Omega : f(p) - \gamma_1 \leq x \leq f(p) + \gamma_2 \} \) containing \( p \). A point which is in this neighborhood for small values of \( \gamma_1 \) and \( \gamma_2 \) is “close” to \( p \). Note that points \( p \) and \( p' \) in Figure 2 are “close” in this sense, but points \( p \) and \( q' \) in Figure 1 remain far apart. We show that if \( p \) destroys a persistent homology group in \( f \), then the neighborhood of \( p \), \( \{ x \in \Omega : f(p) - \gamma_1 \leq x \leq f(p) + \gamma_2 \} \), contains a point \( q \) which destroys a persistent homology group in \( \hat{f} \). The values of \( \gamma_1 \) and \( \gamma_2 \) depend upon the persistence of the homology group and the difference \( \delta \) between \( f \) and \( \hat{f} \).

Theorem 1, one of our main results, states that every destroying critical point of \( f \) contains in its “neighborhood” a similar destroying critical point of \( \hat{f} \). However, to construct a matching of critical points, we need each destroying critical point of \( f \) to be in only one such neighborhood. We establish this stronger result for critical points which are local maxima of functions on manifolds. More specifically, we show that the neighborhoods of local maxima with large persistence are pairwise disjoint.
2 Definitions and assumptions

2.1 Homology groups

For a topological space $X$, the $k$th homology group $H_k(X)$ is an algebraic encoding of the connectivity of $X$ in the $k$th dimension. For a good exposition on homology groups we refer to Hatcher [6]. We will use the singular homology. Although homology groups are defined for coefficients drawn from any ring, we will consider only fields such as $\mathbb{R}, \mathbb{Q}, \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$ as in the previous works [5, 9]. As discussed in Carlsson and Zomorodian [9], computing the persistent homology groups over non-fields is an unsolved and perhaps intractable problem. Over fields the homology groups are vector spaces and the rank of $H_k(X)$, denoted $\beta_k(X)$, is called the $k$th Betti number of $X$.

A continuous map $f : X \to Y$ between two topological spaces $X$ and $Y$ induces a homomorphism, say $f_k$, between their homology groups, $H_k(X) \xrightarrow{f_k} H_k(Y)$. This property is carried over the composition of maps, that is, $(f \circ g)_k = f_k \circ g_k$. In our case, the maps between spaces will be inclusions maps, maps induced by the inclusion map $\iota : X \to Y$. From now on, we take the liberty of dropping the subscript $k$ from $H_k(X)$ when it is clear from the context. For $X \subseteq Y$ the relative homology group of $Y$ with respect to $X$ is given by $H(Y, X) = H(Y) / \ell_\iota(H(X))$ where $H(X) \xrightarrow{\ell_\iota} H(Y)$ is the homomorphism induced by inclusion $\iota : X \to Y$.

A sequence of groups $G_i$ connected by homomorphisms form an exact sequence if any two consecutive homomorphisms in the sequence

$$\ldots \xrightarrow{\ell_i} G_i \xrightarrow{\ell_{i+1}} G_{i+1} \xrightarrow{\ell_{i+2}} \ldots$$

satisfy the property that

$$\text{Im} \, \ell_i = \text{Ker} \, \ell_{i+1}.$$ 

We will use a specific type of sequence called Mayer-Vietoris sequence which is known to be exact.

Let $A, B \subseteq X$ so that $X$ is the union of the interiors of $A$ and $B$ and $D = A \cap B$. The sequence

$$H_k(D) \xrightarrow{\phi} H_k(A) \oplus H_k(B) \xrightarrow{\psi} H_k(X) \xrightarrow{\partial} H_{k-1}(D)$$

is exact and is called the Mayer-Vietoris sequence [6, p. 149]. The map $\partial$ is the connecting homomorphism given by boundary maps [6, p. 116].

2.2 Notation

We use the following notation to define the region bounded by $f^{-1}(a)$ and $f^{-1}(b)$. For $a, b \in \mathbb{R}$ and functions $f$ and $g$, let

$$F^b_a = \{ x \in \Omega : a < f(x) < b \} \text{ and } G^b_a = \{ x \in \Omega : a < g(x) < b \}.$$ 

In our results and proofs we need the space $F^b_a$ and $G^b_a$ closed at the bottom. So, we define

$$\overline{F}^b_a = \{ x \in \Omega : a \leq f(x) < b \} \text{ and } \overline{G}^b_a = \{ x \in \Omega : a \leq g(x) < b \}.$$ 

Notice that $a$ could be $-\infty$ and $b$ could be $\infty$.

2.3 Destruction

Let $\Omega$ be a topological space. For $X \subseteq \Omega$ and $Y \subseteq \Omega$, set $Y$ destroys non-zero $h \in H(X)$ if the image of $h$ under the mapping $H(X) \to H(X \cup Y)$ is zero. In particular, if $q$ is a point in $\Omega$, point $q$ destroys non-zero $h \in H(X)$ if the image of $h$ under the mapping $H(X) \to H(X \cup \{q\})$ is zero.

If $X \subseteq Z \subseteq \Omega$ and $Y \subseteq \Omega$, then we say that $Y$ destroys the image of $h_x \in H(X)$ in $H(Z)$, if $h_z \in H(Z)$ is the image of $h_x$ under the mapping $H(X) \to H(Z)$ and $Y$ destroys $h_z$. We encounter this situation repeatedly where $X$ is some level set $f^{-1}(a)$ and $Y$ is a point. For brevity, we say that point $q$ destroys $h \in f^{-1}(a)$ if point $q$ destroys the image of $h$ in $H(E^{f^{-1}(a)}_q)$.

A function $f : \Omega \to \mathbb{R}$ is point destructible if whenever $h \in H(E^{f^{-1}(a)}_q)$ is destroyed by $f^{-1}(b)$, then $h$ is destroyed by some point $q \in f^{-1}(b)$.

2.4 Maps and spaces

We will be dealing with continuous functions on a compact, connected topological space, $\Omega$. We need some conditions that these functions will be well-behaved, i.e. have properties similar to Morse functions. However, we do not want to restrict ourselves to differentiable functions or to Morse functions.

For a function $f : \Omega \to \mathbb{R}$ and $a \in \mathbb{R}$, let $N_\varepsilon(f^{-1}(a)) = \{ x \in \Omega : a - \varepsilon < f(x) < a + \varepsilon \}$ denote the open $\varepsilon$-neighborhood of $f^{-1}(a)$. The first property we require is that the topology of $f^{-1}(a)$ is similar (isotopic) to the topology of a $\varepsilon$-neighborhood of $f^{-1}(a)$ for suitably small $\varepsilon$. The second property is that $f$ is point destructible. These properties are similar to the Morse condition that critical points are isolated. We define the first property more formally below.

Represent the unit interval $[0, 1]$ by $I$. Subspace $X \subseteq Y$ is a strong deformation retract of $Y$ if there is a continuous $\phi : X \times I \to X$ such that $\phi(y, 0) = y$ and $\phi(y, 1) \in X$ for all $y \in Y$ and $\phi(x, t) = y$ for all $x \in X$ and $t \in \mathbb{R}$. In other words, $\phi$ continuously deforms $Y$ into $X$ without moving any points in $X$. If
$X$ is a strong deformation retract of $Y$, then $H(X)$ is isomorphic to $H(Y)$.

We say that the continuous function $f : \Omega \to \mathbb{R}$ is LR (locally retractable) if for all $a \in \mathbb{R}$, there exists some $\epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$, the set $f^{-1}(a)$ is a strong deformation retract of $N(f^{-1}(a)) = \{ x : a - \epsilon < f(x) < a + \epsilon \}$. Piecewise linear functions on finite simplicial complexes are LR. If a continuous function is Morse, then the function is LR. (See Milnor [7, pp. 12–20] for a proof. Milnor actually proves that $\{ x : f(x) \leq a \}$ is a deformation retract of $\{ x : f(x) \leq a + \epsilon \}$ but his proof also shows that $f^{-1}(a)$ is a strong deformation retract of $\{ x : a - \epsilon < f(x) < a + \epsilon \}$.) Some properties of LR functions will be used in our proofs. We state them in Lemma 1 and Lemma 2. The proofs are given in the appendix.

Lemma 1 says that if $f$ is LR, then we can replace $\{ x : a \leq f(x) \leq b \}$ by suitably chosen small neighborhoods without changing its homology.

**Lemma 1.** If continuous function $f : \Omega \to \mathbb{R}$ is LR, then for every $a, b \in \mathbb{R}$ where $a < b$, there exists an $\epsilon_0$ such that for all $\epsilon \leq \epsilon_0$, set $\{ x : a \leq f(x) \leq b \}$ is a strong deformation retract of $\{ x : f(x) \leq b + \epsilon \}$ and set $\{ x : a \leq f(x) < b \}$ is a strong deformation retract of $\{ x : a - \epsilon < f(x) < b \}$.

Let non-zero $h \in H(f^{-1}(a))$ be destroyed by $F^\infty_a$. If $f$ is LR, then $h$ is destroyed by $\{ x : a \leq f(x) \leq b \}$ for some $b \geq a$. Equivalently, the image of $h$ in $H(E^b)$ is destroyed by $f^{-1}(b)$.

**Lemma 2.** Let $f : \Omega \to \mathbb{R}$ be a continuous, LR function. For any non-zero $h \in H(f^{-1}(a))$, if $H(E^\infty_a)$ destroys $h$, then for some $b \geq a$, the image of $h$ under the mapping $H(f^{-1}(a)) \to H(E^b)$ is destroyed by $f^{-1}(b)$.

Lemma 2 has the following corollary.

**Corollary 3.** Let $f : \Omega \to \mathbb{R}$ be a continuous, point destructible, LR function. For any non-zero $h \in H(f^{-1}(a))$, if $H(E^\infty_a)$ destroys $h$, then some $q \in \Omega$ destroys $h$.

**Proof.** By Lemma 2, there exists some $b > a$ such that the image $h'$ of $h$ under the mapping $H(f^{-1}(a)) \to H(E^b)$ is destroyed by $f^{-1}(b)$. Since $f$ is point destructible, there is some point $q \in f^{-1}(b)$ which destroys $h'$ and $h$.

# 3 Persistence

Intuitively, the persistence of a point $p \in \Omega$ is the “age” of the “oldest” homology element destroyed by $p$. Formally, the persistence of point $p \in \Omega$ is

$$\Pi_f(p) = \limsup \{ f(p) - a : p \text{ destroys some non-zero } h \in H(f^{-1}(a)) \}.$$

We use $\limsup$ in place of $\max$ because it is possible that $p$ destroys non-zero elements of $H(f^{-1}(a + \epsilon))$ for any $\epsilon > 0$ but not elements of $H(f^{-1}(a))$.

If point $p$ has persistence $\gamma$, and $f(p) - \gamma < a < f(p)$, does $p$ destroy some element of $H(f^{-1}(a))$? As we show below, the answer is yes.

We will now use subsets of $\Omega$ which are bounded by two different functions in our discussion of persistence. The set $X = \{ x : a \leq f(x) < b \}$ is properly bounded by $f$ and $g$ at $a$ and $b$ respectively, if the level sets $f^{-1}(a)$ and $g^{-1}(b)$ are disjoint and $X$ is non-empty. Notice that $X$ includes $f^{-1}(a)$ at the bottom but not $g^{-1}(b)$ at the top.

In Lemma 4 we will establish a result relating three functions and the spaces delimited by their level sets. Later we will set these level sets only to those of two functions $f$ and $g$ that are used to establish the stability result.

Let $f_1, f_2, f_3 : \Omega \to \mathbb{R}$ be three continuous LR functions. Let $q$ be a point in $\Omega$ and let $a_3 = f_3(q)$. Let $X = \{ x \in \Omega : a_1 \leq f_1(x) < a_2 \}$ and $X' = \{ x \in \Omega : a_2 \leq f_2(x) < a_3 \}$ be properly bounded subsets of $\Omega$ where $X' \subseteq X$ and $f_1^{-1}(a_1) \cap f_2^{-1}(a_2) = \emptyset$, see Figure 3. The essence of the following lemma is that if $q$ destroys a generator in $H(f_3^{-1}(a_1))$, then it necessarily destroys some generator in $H(f_2^{-1}(a_2))$. Some subtle aspect of this destruction is narrated in the caption of Figure 3.

![Figure 3](image-url)

Figure 3: Sets $X = \{ x \in \Omega : a_1 \leq f_1(x) < a_2 \}$ and $X' = \{ x \in \Omega : a_2 \leq f_2(x) < a_3 \}$. Cycle $C$ generates $h_x \in H(X)$ and $h_x' \in H(X')$ where both $h_x$ and $h_x'$ are destroyed at $q$. Elements $h_x$ and $h_x'$ are the images of some $h_1 \in H(f_1^{-1}(a_1))$ and some $h_2 \in H(f_2^{-1}(a_2))$, respectively. The mapping $H(X') \to H(X)$ sends $h_x'$ to $h_x$. Cycle $C''$ generates $h_x'' \in H(X)$ and $h_x''' \in H(X')$ where $h_x''$ is destroyed at $q$ but $h_x'''$ is not, even though the mapping $H(X') \to H(X)$ sends $h_x''$ to $h_x$. 
Lemma 4. If \( q \) destroys some non-zero \( h_x \in H(X) \) and \( h_x \) is the image of some \( h_1 \in H(f^{-1}(a_1)) \), then \( h_x \) is the image of some non-zero \( h'_x \in H(X') \) which is destroyed by \( q \). Moreover, \( h_x \) is the image of some non-zero \( h_2 \in H(f^{-1}(a_2)) \).

Proof. Let \( h_x \in H(X) \) be the image of some \( h_1 \in H(f^{-1}(a_1)) \) under the mapping \( H(f^{-1}(a_1)) \rightarrow H(X) \) where \( h_x \) is destroyed by \( q \). By Mayer-Vietoris, the sequence

\[
H(X') \rightarrow H(X' \cup \{q\}) \oplus H(X) \rightarrow H(X \cup \{q\})
\]

is exact. Since the induced mapping \( H(X) \rightarrow H(X \cup \{q\}) \oplus H(X) \rightarrow H(X \cup \{q\}) \) sends \( h_x \) to 0, the mapping \( H(X' \cup \{q\}) \oplus H(X) \rightarrow H(X \cup \{q\}) \) sends \( (0 \oplus h_x) \) to 0. Since the sequence is exact, there is some \( h'_x \), whose image is \( 0 \oplus h_x \) under the mapping \( H(X) \rightarrow H(X \cup \{q\}) \oplus H(X) \). Thus \( h_x \) is the image of \( h'_x \) and \( h'_x \) is destroyed by \( q \).

We now prove that \( h'_x \) is the image of some \( h_2 \in H(f_2^{-1}(a_2)) \). Since \( f_2 \) is LR, there exists some \( \epsilon_1 > 0 \) such that \( H(f_2^{-1}(a_2)) \rightarrow H(N_{\epsilon}(f_2^{-1}(a_2))) \) is an isomorphism for all \( \epsilon' \leq \epsilon_1 \). By Lemma 1, \( H(X') \rightarrow H(X' \cup N_{\epsilon}(f_2^{-1}(a_2))) \) is an isomorphism for all \( \epsilon' \leq \epsilon_2 \). Since \( f_2^{-1}(a_1) \cap f_2^{-1}(a_2) = \emptyset \), there is some \( \epsilon_3 \) such that \( N_{\epsilon_3}(f_2^{-1}(a_2)) \subseteq H(X) \). Let \( \epsilon \) be the smaller of \( \epsilon_1 \), \( \epsilon_2 \), and \( \epsilon_3 \).

Let \( Y \) equal \( N_{\epsilon}(f_2^{-1}(a_2)) \). Let \( Z = \{ x \in \Omega : a_1 \leq f_1(x) \text{ and } f_2(x) < a_2 + \epsilon \} \) and \( Z' = \{ x \in \Omega : a_2 - \epsilon < f_2(x) \text{ and } f_3(x) < a_3 \} \). The following commutative diagram gives the relevant mappings between homology groups:

\[
\begin{array}{ccc}
H_1 & \xrightarrow{h_1} & H_1' \\
\downarrow h_2 & & \downarrow h_2' \\
H_2 & \xrightarrow{h_2} & H_2' \\
\downarrow h_y & & \downarrow h_y' \\
H(Y) & \xrightarrow{h_y} & H(Z) \\
H(Z') & \xrightarrow{h_z} & H(X) \end{array}
\]

Element \( h_x \) is the image of \( h_1 \in H(f_1^{-1}(a_1)) \). Let \( h_z \) be the image of \( h_1 \) under the mapping \( H(f_1^{-1}(a_1)) \rightarrow H(Z) \). Let \( h'_z \) be the image of \( h_z' \) under the mapping \( H(X') \rightarrow H(Z') \). Element \( h_z' \) is the image of both \( h_z \) and \( h'_z \) under the respective mappings \( H(Z) \rightarrow H(X) \) and \( H(Z') \rightarrow H(X) \).

By Mayer-Vietoris, the sequence

\[
H(Y) \rightarrow H(Z) \oplus H(Z') \rightarrow H(X)
\]

is exact. Since the mapping \( H(Z) \oplus H(Z') \rightarrow H(X) \) sends \((h_z \oplus h'_z)\) to \((h_x - h_z) = 0\), element \((h_z \oplus h'_z)\) must be in the image of some \( h_y \in H(Y) \). Since \( H(f_2^{-1}(a_2)) \rightarrow H(Y) \) is an isomorphism, there is some \( h_y \in H(f_2^{-1}(a_2)) \) whose image is \( h_y \) under the mapping \( H(f_2^{-1}(a_2)) \rightarrow H(Y) \).

All the mappings of homology groups are induced by the inclusion mapping and thus the diagram is commutative. Since \( H(X') \rightarrow H(Z') \) is an isomorphism, the inverse mapping \( H(Z') \rightarrow H(X') \) takes \( h'_z \) to \( h_z' \). The mapping \( H(f_2^{-1}(a_2)) \rightarrow H(Y) \rightarrow H(Z') \rightarrow H(X') \) sends element \( h_2 \in H(f_2^{-1}(a_2)) \) to \( h'_z \in H(X') \). Thus \( h'_z \in H(X') \) is the image of some non-zero \( h_2 \in H(f_2^{-1}(a_2)) \).

Setting \( f_1 = f_2 = f_3 \) gives the following corollary:

**Corollary 5.** Let \( f : \Omega \rightarrow \mathbb{R} \) be a continuous, LR function. If \( q \in \Omega \) has persistence \( \gamma \), then for every \( a \) where \( f(a) - \gamma < a < f(q) \), point \( q \) destroys some element of \( H(f^{-1}(a)) \).

### 4 Stability

In this section we prove one of our main results, Theorem 1. Let \( f \) and \( g \) be two functions defined on \( \Omega \). We say \( |f - g| < \delta \) if \( |f(x) - g(x)| < \delta \) for all \( x \in \Omega \).

We show that if \( q \) is a destructor for \( f \) with persistence \( \gamma > 2\delta \), there is a point \( q' \) which is a destructor for \( g \) where \( q \) and \( q' \) lie in the same connected component of the space \( \mathbb{F}_{Q(q)} \). (Wherever we use the term connected component, we always mean path connected.)

Moreover, the values \( f(q) \) and \( g(q') \) are close. This theorem not only relates \( q \) and \( q' \) in the range as in Cohen-Steiner et al. [3] but also in the domain.

We will need the concept of chains and cycles that define the homology groups. Chains are formal sums of maps from standard simplices into the domain \( \Omega \) and cycles are chains which have no boundary. The boundary of a chain is always a cycle. For details, see Hatcher [6].

**Lemma 6.** If non-zero \( h \in H(F^n) \) is destroyed by point \( q \) and cycle \( C \subseteq F^n \) generates \( h \), then \( C \) and \( q \) lie in the same connected component of \( \text{cl}(F^n) \).

Proof. Since \( q \) destroys \( h \), cycle \( C \) is the boundary of some chain \( D \subseteq F^n \cup \{q\} \). The chain \( D \) must contain point \( q \) or else \( C \) would be the boundary of a chain in \( F^n \) and \( h \) would be 0. Since all points in \( D \) other than \( q \) lie in \( F^n \), point \( q \) is in \( \text{cl}(F^n) \). Since \( D \) connects \( C \) and \( q \) in \( \text{cl}(F^n) \), cycle \( C \) and point \( q \) lie in the same component of \( \text{cl}(F^n) \).

**Theorem 1.** Let \( f, g : \Omega \rightarrow \mathbb{R} \) be continuous, point destructible, LR functions on \( \Omega \) where \( |f - g| < \delta \). If \( q \) destroys non-zero \( h \in H(f^{-1}(a)) \) and \( |f(q) - a| > 2\delta \),
and $\sigma$ is the connected component of $\text{cl}((F_{\delta}^{b})^{+} +g\delta)$ containing $q$, then $\sigma$ contains a point $q'$ for $g$ which destroys some $h_\delta \in H(g^{-1}(a+\delta))$ and $f(q) - \delta \leq g(q') \leq f(q) + \delta$.

**Proof.** Let $b$ equal $f(q)$. Let

$$X = \{x \in \Omega : a + \delta \leq g(x) \text{ and } f(x) < b\}.$$  

Since $|f(x) - g(x)| < \delta$ and $b-a > 2\delta$, space $X$ is properly bounded. Note that $F_{\delta}^b \subseteq X \subseteq F_{\delta}^b$ and so there exist homomorphisms $H(G_{\alpha+\delta}^{b-\delta}) \rightarrow H(X) \rightarrow H(F_{\delta}^b)$ induced by inclusions.

The following commutative diagram gives the relevant mappings between homology groups:

$$
\begin{array}{ccc}
  h_\delta & \in & H(g^{-1}(a+\delta)) \\
  & \downarrow & \\
  H(G_{\alpha+\delta}^{b-\delta}) & \rightarrow & H(X) \rightarrow H(F_{\delta}^b) \ni h_f' \\
  & \downarrow & \\
  h'_g & \in & H(G_{\alpha+\delta}^{b'}+\delta) \\
  & \downarrow & \\
  H(X \cup \{q\}) & \rightarrow & H(F_{\delta}^b \cup \{q\}) \\
  & \downarrow & \\
  H(G_{\alpha+\delta}^{\infty}) & \ni & H(G_{\alpha+\delta}^{b'+\delta})
\end{array}
$$

The value $b'$ will be defined below.

Let $h_f' \in H(F_{\delta}^b)$ be the image of $h_f$ under the mapping $H(f^{-1}(a)) \rightarrow H(F_{\delta}^b)$. Since $h_f'$ is destroyed by $f^{-1}(b)$, element $h'_f$ is non-zero. By Lemma 4, element $h_f'$ is the image of some element $h_x \in H(X)$ which is destroyed by $q$ and is the image of some $h_\delta \in H(g^{-1}(a+\delta))$. Since $h_f'$ is non-zero, elements $h_x$ and $h_\delta$ are non-zero.

Since $h_x$ is destroyed by $q$, the mapping $H(X) \rightarrow H(X \cup \{q\}) \rightarrow H(G_{\alpha+\delta}^{\infty})$ sends $h_x$ to zero. Thus the composition of mappings $H(g^{-1}(a+\delta)) \rightarrow H(X) \rightarrow H(G_{\alpha+\delta}^{\infty})$ sends $h_\delta$ to $h_x$ to zero.

By Corollary 3, there exists a point $q' \in \Omega$ such that $h_g$ is destroyed by $q'$ (i.e., the image of $h_g$ under the mapping $H(g^{-1}(a+\delta)) \rightarrow H(G_{\alpha+\delta}^{\infty})$ is destroyed by $q'$). Let $b'$ equal $f(q')$. Since $G_{\alpha+\delta}^b$ is a subset of $X$, the image of $h_g$ under the mapping $H(g^{-1}(a+\delta)) \rightarrow H(G_{\alpha+\delta}^{\infty})$ is non-zero and so $b' \geq b-\delta$. Let $h_g'$ be the image of $h_g$ under the mapping $H(g^{-1}(a+\delta)) \rightarrow H(G_{\alpha+\delta}^{\infty})$.

Element $h_g$ is generated by some cycle $C$ in $g^{-1}(a+\delta)$. Since the image of $h_g$ is $h_x \in H(X)$ under the mapping $H(g^{-1}(a+\delta)) \rightarrow H(X)$, cycle $C$ also generates $h_x$. Similarly, cycle $C$ generates $h_g' \in H(G_{\alpha+\delta}^{b'})$ and $h_g' \in H(F_{\delta}^b)$.

Since $h_x$ is destroyed by $q$, cycle $C$ is the boundary of some chain $D \subseteq X \cap f^{-1}(b)$. Since $|f-g| \leq \delta$, set $X \cup f^{-1}(b)$ is a subset of $G_{\alpha+\delta}^{b'\delta} \cup g^{-1}(b+\delta)$ and so chain $D$ is a subset of $G_{\alpha+\delta}^{b'\delta} \cup g^{-1}(b+\delta)$. Since $h_g' \in H(G_{\alpha+\delta}^{b'})$ is non-zero, space $G_{\alpha+\delta}^{b'\delta} \cup g^{-1}(b+\delta)$ cannot be a subspace of $G_{\alpha+\delta}^{b'\delta}$. Thus $b'$ is at most $b+\delta$.

As noted above, cycle $C$ generates $h_g' \in H(G_{\alpha+\delta}^{b'})$ and $h_g' \in H(F_{\delta}^b)$. Point $q$ destroys $h_g'$ which is generated by $C$. By Lemma 6, point $q$ must lie in the connected component of $\text{cl}(F_{\delta}^b)$ containing $C$. Similarly, since $q'$ destroys $h_g'$, point $q'$ must lie in the connected component of $\text{cl}(F_{\delta}^b)$ containing $C$. Since $F_{\delta}^b \subseteq \text{cl}(F_{\delta}^{b+2\delta})$ and $\text{cl}(F_{\delta}^{b+\delta}) \subseteq \text{cl}(F_{\delta}^{b'\delta}) \subseteq \text{cl}(F_{\delta}^{b+2\delta})$, points $q$ and $q'$ must lie in the same connected component $\sigma$ of $\text{cl}(F_{\delta}^{b+2\delta})$. □

## 5 Computing persistence

Theorem 1 can be used to compare two real valued functions $f$ and $g$ defined on a topological space $\Omega$. The key computation to apply Theorem 1 is:

(i) determine if a point $p$ which destroys some $h \in H(f^{-1}(a))$ has persistence greater than $\gamma$.

We use Betti numbers and their persistent counterparts to compute (i). Recall that, for a homology group $H(X)$, the Betti number is $\beta(X) = \dim H(X)$. It gives the number of generators in $H(X)$. The persistent Betti numbers relate the homology classes of one space into the other. For $Y \subseteq X$, let $H^*_Y$ be the image of the map $H(Y) \rightarrow H(X)$ induced by inclusion $Y \rightarrow X$. Define $\beta(Y,X) = \dim H^*_X$. In words, $\beta(Y,X)$ counts the
number of non-zero generators of $H(Y)$ that remain so in the larger space $X$.

We discuss the computations for the function $f$. It is clear that similar computations are needed for $g$ as well. In general, for a point $p$ and a value $a < b = f(p)$ we want to compute if an element of $H(f^{-1}(a))$ gets destroyed by $p$. Let

$$
\beta^b_a = \beta(f^{-1}(a), F^b_a) \quad \text{and} \quad \lambda^b_a = \beta(f^{-1}(a), F^b_a \cup \{p\}).
$$

Note that $p$ is a point whereas $a$ and $b$ are real values. The number $\beta^b_a$ counts the number of generators of $H(f^{-1}(a))$ surviving in $F^b_a$ and $\lambda^b_a$ counts the number of generators of $H(f^{-1}(a))$ surviving in $F^b_a \cup \{p\}$. Therefore,

$$
\pi^b_a = \beta^b_a - \lambda^b_a
$$

counts the number of generators of $H(f^{-1}(a))$ destroyed by $p$. So, if $\pi^b_a > 0$, we have a generator of $H(f^{-1}(a))$ that is destroyed by $p$ where $f(p) = b$.

Notice that Theorem 1 can be applied to any point $p \in \Omega$ and any value $a < f(p)$. However, for a canonical computation one can focus on the critical points of the functions. We define the critical points of $f$ as follows.

**Definition 1.** A point $p \in \Omega$ is critical for $f : \Omega \to \mathbb{R}$ if $H(F^b_a) \rightarrow H(F^b_a \cup \{p\})$ is not an isomorphism for some $a, b \in \mathbb{R}$ where either $f(p) = a$ or $f(p) = b$.

The above definition of critical points is similar to that of Cohen-Steiner et al. [3] with a distinction that the relevant space is an interval set whose lower level may not be at $-\infty$. Also, notice that the space $F^b_a$ could be above or below the level of $f(p)$.

Compute the critical points of $f$ according to the above definition. Let $p_0, p_1, \ldots, p_k$ be the critical points of $f$ ordered according to the increasing values, that is, $f(p_i) > f(p_{i-1})$ for all $i \geq 0$. We compute the persistence $\Pi_f(p_i)$ for these critical points $p_i$ as follows. For $1 \leq i < k - 1$, let $a_i$ be a value with $f(p_{i-1}) < a_i < f(p_i)$. Compute $\pi^{a_i}_{p_i}$ for any pair $i, j$ where $j \geq i > 0$. Since $\pi^{a_i}_{p_i}$ is constant for all $a$ where $f(p_{i-1}) < a < f(p_i)$, if $\pi^{a_i}_{p_i}$ is greater than 0, then the persistence $\Pi_f(p_i)$ is at least $f(p_i) - f(p_{i-1})$. Thus we compute $\Pi_f(p_i)$ as

$$
\max_{i} [f(p_i) - f(p_{i-1})] \quad \text{so that} \quad \pi^{a_i}_{p_i} > 0.
$$

Similarly, we can compute the critical points $q_0, q_1, \ldots, q_n$ and a set of intermittent values $b_1, b_2, \ldots, b_{n-1}$ for the function $g$. The persistence of a critical point $q$ of $g$ is measured similarly by $\Pi_g(q)$.

To compare $f$ and $g$, we check if any critical point $p$ of $f$ has $\Pi_f(p)$ greater than a user supplied parameter $\tau$. If so, we search for a critical point $q$ of $g$ in the connected component of $\text{cl}(F^f_{f(p)+\tau})$ so that $\Pi_g(q) > \tau$ and $|f(p) - g(q)| \leq \frac{\tau}{2}$. If $\tau > 2\delta$, such a $q$ exists by Theorem 1.

### 5.1 PL case

Assume there is some finite triangulation of $\Omega$ such that $f$ and $g$ are linear on each simplex of the triangulation. Functions $f$ and $g$ are LR (locally retractable), but not necessarily point destructible. The critical points of $f$ and $g$ are located at the triangulation vertices. A small perturbation of the scalar value at each triangulation vertex and the linear interpolation of those values over the triangulation simplices, gives new piecewise linear functions which are point destructible.

Carlsson and Zomorodian in [9] show how to compute persistent Betti numbers for homology groups of filtered simplicial complexes over any field. However, spaces $F^b_a$ and $F^b_a \cup \{p\}$ are not closed. To compute their persistent Betti numbers, $\beta^b_a$ and $\lambda^b_a$, we collapse them onto closed sets which are simplicial complexes.

Consider the space $F^b_a$. Let $t \subset \Omega$ be a simplex with a point in this space. If each vertex $v$ of $t$ either has $f(v) \geq b$ or $a \leq f(v) < b$, the subset $t \cap F^b_a$ can be collapsed to the face of $t$ made by the vertices whose values lie in $[a, b)$. This cannot be done for simplices that cut across the levels of $a$ and $b$. These simplices have vertices with values above $b$ and also below $a$. For such a simplex we take an edge $e = \{u, v\}$ where $f(u) < a$ and $f(v) \geq b$ and consider a point $x$ on this edge where $a < f(x) < b$. We divide $t$ by starring from $x$ to all its vertices. After subdividing all such simplices we obtain a subdivision $\Omega$ of $\Omega$ which has no simplex cutting across the interval $[a, b]$. Consider the simplicial complex made by the collection of simplices in $\Omega$ that have all vertices with values in $[a, b]$. The underlying space of this simplicial complex is a deformation retract of $F^b_a$ and therefore has homology groups isomorphic to that of $F^b_a$.

### 6 Maxima

We show that the neighborhoods of local maxima with large persistence are pairwise disjoint. This enables us to establish a matching of such critical points.

The idea of the proof is as follows. Consider two local maxima, $p, p' \in \Omega$, where $f(p) \leq f(p')$. If $p$ destroys non-zero $h \in H_k(f^{-1}(a))$, then $k$ equals $d - 1$. Let $c$ be the connected component of $F^\infty$ containing $p$. If $c$ is a manifold with boundary, then $F^f_{f(p')} \cup \{p\}$ must contain $c$. Since $f(p) \leq f(p')$, set $F^f_{f(p')} \cup \{p\}$ does not contain $p'$ and therefore point $p'$ is not in $c$. 


Lemma 7. Let $E_\infty$ be an oriented $d$-manifold with boundary. If $p$ is a local maximum and $p$ destroys non-zero $h \in H_k(E_\alpha(p))$, then $k$ equals $d - 1$.

Proof. See appendix.

As previously noted, we always mean path connected when referring to connected components.

Lemma 8. Let $M$ be a connected, oriented $d$-manifold with non-empty boundary. Let $D_1, D_2, \ldots, D_k$ be the connected components of $\partial M$ with orientation inherited from $M$. If $a_1D_1 + a_2D_2 + \ldots + a_kD_k$ generate the zero element of $H_{d-1}(M)$, then $a_1 = a_2 = \ldots = a_k$.

Proof. The sequence $H_d(M) \to H_d(M, \partial M) \to H_{d-1}(\partial M) \to H_{d-1}(M)$ is exact [6, Theorem 2.16, p. 117]. Since the boundary of $M$ is not empty, the homology group $H_d(M)$ is zero. The homology group of $H_d(M, \partial M)$ is $\mathbb{Z}$, the ground ring of the homology group. The map $H_d(M, \partial M) \to H_{d-1}(\partial M)$ is the connecting homomorphism. It maps $\mathbb{Z}$ to $H_{d-1}(\partial M)$ which is generated by $D_1 + D_2 + \ldots + D_k$. Since the mapping is exact, the image of $\mathbb{Z}$ under the mapping $H_{d-1}(\partial M) \to H_{d-1}(M)$ is zero. Moreover, only elements in $\mathbb{Z}$ map to zero. Thus, if $a_1D_1 + a_2D_2 + \ldots + a_kD_k$ generate the zero element of $H_{d-1}(M)$, then $a_1 = a_2 = \ldots = a_k$.

Lemma 9. Let $M$ be a connected, oriented $d$-manifold with non-empty boundary. If $\partial M \subseteq M^I \subseteq M$ and $H_{d-1}(\partial M) \to H_{d-1}(M^I)$ takes non-zero $h \in H_{d-1}(\partial M)$ to zero, then $M^I$ equals $M$.

Proof. Assume $M^I$ does not equal $M$. Let $p$ be a point in $M - M^I$. Let $B$ be an open topological ball containing $p$ whose closure does not intersect $\partial M$. There exists a deformation retract from $M - \{p\}$ to $M - B$. Thus $H(M - \{p\})$ is isomorphic to $H(M - B)$.

The mapping $H_{d-1}(\partial M) \to H_{d-1}(M - \{p\}) \to H_{d-1}(M - B)$ sends $h$ to zero in $H_{d-1}(M - B)$. Let $h_0$ be the element of the homology group of $H_{d-1}(\partial M)$ generated by $\partial M$ with orientation inherited from $M$. By Lemma 8, element $h$ equals $\alpha h_0$ for some non-zero $\alpha$. Let $h_B$ be the element of $H_{d-1}(M - B)$ generated by $\partial B$ with orientation inherited from $M - B$. Let $h'$ be the image of $h$ under the map $H_{d-1}(\partial M) \to H_{d-1}(\partial M \cup \partial B)$. The element $h$ and hence $h'$ is sent to zero in $H_{d-1}(M - B)$. By Lemma 8, element $h'$ equals $\beta(h_0 + h_B)$ for some non-zero $\beta$. Thus $\alpha h_0 = \beta(h_0 + h_B)$. Since $h_0$ and $h_B$ are linearly independent, $\alpha$ and $\beta$ are both zero implying $h$ is a zero element, a contradiction. It follows that $M$ equals $M^I$.

Let $\sigma^f_p(\gamma)$ represent the connected component of $E_\infty(p) - \gamma$ containing $p$. We prove that the neighborhoods $\sigma^f_p(\gamma)$ of points with persistence greater than $\gamma$ are pairwise disjoint. (See Figure 5.)

Theorem 2. Let $f : \Omega \to \mathbb{R}$ be a continuous function such that $E_\infty$ is a $(d - 1)$-manifold with boundary for all but a finite number of $a$. If points $p_0, p_1 \in \Omega$ are local maxima with persistence greater than $\gamma$, then $\sigma^f_{p_0}(\gamma)$ does not intersect $\sigma^f_{p_1}(\gamma)$.

Proof. Let $p_0, p_1 \in \Omega$ be local maxima with persistence $\gamma_0, \gamma_1$, both greater than $\gamma$. Without loss of generality, assume that $f(p_0) \leq f(p_1)$.

Assume that $\sigma^f_{p_0}(\gamma)$ intersects $\sigma^f_{p_1}(\gamma)$. Since $E_\infty$ is a $(d - 1)$-manifold for all but a finite number of $a$, there is some $\gamma' \geq \gamma$ such that $\gamma_0 > \gamma'$ and $E_\infty(f(p_0) - \gamma')$ is a $(d - 1)$-manifold with boundary. Since $\sigma^f_{p_0}(\gamma)$ intersects $\sigma^f_{p_1}(\gamma)$, set $\sigma^f_{p_0}(\gamma')$ intersects $\sigma^f_{p_1}(\gamma')$.

Since $f(p_0) \leq f(p_1)$, set $E_\infty(f(p_0) - \gamma')$ contains $E_\infty(f(p_1) - \gamma')$. Since $\sigma^f_{p_0}(\gamma')$ intersects $\sigma^f_{p_1}(\gamma')$, set $\sigma^f_{p_0}(\gamma')$ contains $\sigma^f_{p_1}(\gamma')$. Thus $\sigma^f_{p_0}(\gamma')$ contains $p_1$.

By Lemma 4, point $p_0$ destroys some non-zero element of $H_k(f^{-1}(f(p_0) - \gamma'))$. By Lemma 7, $k$ equals $d - 1$. By Lemma 10 (appendix), point $p_0$ destroys a non-zero element of $H_{d-1}(\partial \sigma^f_{p_0}(\gamma'))$. In Lemma 9 putting $M = \sigma^f_{p_0}(\gamma')$ and $M'$ equal the connected component of $E_\infty(f(p_0) - \gamma')$ containing $p_0$, we conclude $E_\infty(f(p_0) - \gamma') \cup \{p_0\}$ contains $\sigma^f_{p_0}(\gamma')$ and thus contains $p_1$. However, since $f(p_0) \leq f(p_1)$, set $E_\infty(f(p_0)) \cup \{p_0\}$ does not contain $p_1$. Thus, $\sigma^f_{p_0}(\gamma)$ does not intersect $\sigma^f_{p_1}(\gamma)$.

Our final theorem gives relationships between neighborhoods of local maxima of $f$ and of $g$.

Theorem 3. Let $f, g : \Omega \to \mathbb{R}$ be continuous functions such that $E_\infty$ and $\mathbb{C}_\infty$ are $(d - 1)$-manifolds with boundary for all but a finite number of $a$ and $|f - g| < \delta$. Let $p \in \Omega$ be a local maximum of $f$ and let $q$ and $q'$ be local maxima of $g$ such that $p, q, q'$ have persistence greater than $\gamma$ and $|f(p) - g(q)| < \delta$ and $|f(p) - g(q')| < \delta$.

(i) If $\sigma^f_p(\gamma - 2\delta)$ intersects $\sigma^g_q(\gamma - 2\delta)$, then $\sigma^f_p(\gamma)$ contains $\sigma^g_q(\gamma - 2\delta)$.

(ii) If $\sigma^g_q(\gamma - 2\delta)$ intersects $\sigma^f_p(\gamma - 2\delta)$, then $\sigma^g_q(\gamma - 2\delta)$ does not intersect $\sigma^f_p(\gamma - 2\delta)$.

Proof of (i). Let $y$ be a point in $\sigma^f_p(\gamma - 2\delta) \cap \sigma^g_q(\gamma - 2\delta)$ and $z$ be any point in $\sigma^g_q(\gamma - 2\delta)$. Set $\sigma^f_p(\gamma - 2\delta)$ is path connected, so there is a path $\zeta \subseteq \sigma^f_p(\gamma - 2\delta)$ from $y$ to $z$.

Since $\zeta \subseteq \sigma^f_p(\gamma - 2\delta)$, $f(x) \geq f(p) - \gamma + 2\delta$ for every point $x \in \zeta$. Since $|f(x) - g(x)| < \delta$ for all $x \in \Omega$ and $|f(p) - g(q)| < \delta$, it follows that $g(x) \geq g(p) - \gamma$ for all
Proof of (ii). By Theorem 2, \( \sigma_q^f(\gamma) \) and \( \sigma_q^g(\gamma) \) are disjoint. By (i) above, if \( \sigma_p^f(\gamma - 2\delta) \) intersects \( \sigma_q^g(\gamma - 2\delta) \), then \( \sigma_p^g(\gamma) \) contains \( \sigma_q^f(\gamma - 2\delta) \). Similarly, if \( \sigma_p^f(\gamma - 2\delta) \) intersects \( \sigma_q^g(\gamma - 2\delta) \), then \( \sigma_p^g(\gamma) \) contains \( \sigma_q^f(\gamma - 2\delta) \). However, \( \sigma_p^g(\gamma) \) and \( \sigma_q^g(\gamma) \) are disjoint. Thus, if \( \sigma_p^f(\gamma - 2\delta) \) intersects \( \sigma_q^g(\gamma - 2\delta) \), then \( \sigma_p^g(\gamma - 2\delta) \) does not intersect \( \sigma_q^g(\gamma - 2\delta) \). (See Figure 5.)

7 Matching

We assume that \( f, g : \Omega \to \mathbb{R} \) are continuous, point destructible, LR functions such that \( E^\infty \) and \( G^\infty \) are \((d - 1)\)-manifolds with boundary for all but a finite number of \( \alpha \). Let \( M_f \) and \( M_g \) be the set of local maxima of \( f \) and \( g \), respectively, and let \( M_f(\gamma) \subseteq M_f \) and \( M_g(\gamma) \subseteq M_g \) be the set of local maxima of \( f \) and \( g \), respectively, with persistence greater than \( \gamma \). We would like to match points in \( M_f(\gamma) \) with close points in \( M_g(\gamma) \) in the sense of Theorem 1. However, there may be no such matching. In fact, \( f \) may contain a set of critical points with persistence a little bit above \( \gamma \) while nearby critical points in \( g \) all have persistence a bit below \( \gamma \). Thus, \( M_f(\gamma) \) can contain any number of points while \( M_g(\gamma) \) is empty! Instead of matching \( M_f(\gamma) \) and \( M_g(\gamma) \) only with each other, we allow them to match with points with slightly less persistence.

We say that a partial matching of \( M_f \) with \( M_g \) covers \( M_f(\gamma) \) and \( M_g(\gamma) \) if all points in \( M_f(\gamma) \) and \( M_g(\gamma) \) are matched. As before, let \( \sigma_p^f(\gamma) \) and \( \sigma_q^g(\gamma) \) be the connected components of \( E^\infty_{\gamma}(p) \) and \( G^\infty_{\gamma}(q) \) containing \( p \) and \( q \), respectively. A partial matching of \( M_f \) and \( M_g \) is \((\alpha, \beta)\)-close if for each pair \((p, q)\) where \( p \in M_f \) and \( q \in M_g \), point \( q \) lies in \( \sigma_p^f(\alpha) \) and point \( p \) lies in \( \sigma_q^g(\alpha) \), and \( |f(p) - g(q)| < \beta \).

Assume that \( |f(x) - g(x)| < \gamma/4 \) for all \( x \in \Omega \). We will find a partial matching of \( M_f \) with \( M_g \) which covers \( M_f(\gamma) \) and \( M_g(\gamma) \) and is \((\gamma, \gamma/4)\)-close.

The algorithm is as follows. For each point \( p \in M_f(\gamma) \), we compute \( \sigma_p = \sigma_p^f(\gamma/2) \). Similarly, for each \( q \in M_g(\gamma) \), we compute \( \sigma_q = \sigma_q^g(\gamma/2) \). By Theorem 3, each \( \sigma_p \) intersects at most one \( \sigma_q \) where \( |f(p) - g(q)| < \gamma/4 \) and vice versa. If \( \sigma_p \) intersects such a \( \sigma_q \), then match \( p \) with \( q \). If not, then match \( p \) with some \( q' \in M_g(\gamma/2) \) lying in \( \sigma_p \) such that \( |f(p) - f(q')| < \gamma/4 \). (By Theorem 1 such a \( q' \) exists.) Similarly, if \( \sigma_q \) does not intersect any \( \sigma_p \), match \( q \) with \( p' \in M_f(\gamma/2) \) lying in \( \sigma_q \) such that \( |f(q') - f(p)| < \gamma/4 \).

We claim that the algorithm \( \text{MatchPersistentMax} \) matches all maxima with persistence more than \( \gamma \):

**Proposition 1.** If \( |f - g| \leq \gamma/4 \), then \( \text{MatchPersistentMax}(\Omega, f, g, \gamma) \) produces a partial matching of \( M_f \) with \( M_g \) which covers \( M_f(\gamma) \) and \( M_g(\gamma) \) such that every matched pair \((p, q)\) where \( p \in M_f \) and \( q \in M_g \) is \((\gamma, \gamma/4)\)-close.

**Proof.** By Theorem 2, the \( \sigma_p = \sigma_p^f(\gamma/2) \), \( p \in M_f(\gamma) \), are pairwise disjoint and the \( \sigma_q = \sigma_q^g(\gamma/2) \), \( q \in M_g(\gamma) \) are pairwise disjoint. By Theorem 4, each \( \sigma_p \) intersects at most one \( \sigma_q \) and vice versa. Thus Step 4 gives a one to one partial matching.

By Theorem 1, \( \sigma_p \) contains some point \( q' \in M_f(\gamma/2) \) such that \( |f(p) - g(q')| \leq \gamma/4 \). Since \( (p, q') \) is not matched in Step 4, point \( q' \) is not in \( M_g(\gamma) \). Thus point \( q' \) is not matched in Step 4. Since \( \sigma_p \) does not intersect any \( \sigma_p' \), \( p' \in M_f(\gamma) \), point \( q' \) is matched to at most one \( p \) in Step 5. Similarly, point \( p' \) in Step 6 is not matched in Steps 4 and 5 and is matched to at most one \( q \) in Step 6. Thus the matching is one to one and covers all of \( M_f(\gamma) \).
and $M_g(\gamma)$.

It remains to show that for each match $(p, q)$, set $\sigma_q^p(\gamma)$ contains $q$ and $\sigma_q^p(\gamma)$ contains $p$. If $p$ and $q$ are matched in Step 4, then $\sigma_q^p(\gamma/2)$ intersects $\sigma_q^p(\gamma/2)$. This holds true even if $p$ and $q$ are matched in Steps 5 or 6. By Theorem 3 with $\delta = \gamma/4$, $\sigma_q^p(\gamma)$ contains $\sigma_q^p(\gamma/2)$ which contains $q$ and $\sigma_q^p(\gamma)$ contains $\sigma_q^p(\gamma/2)$ which contains $p$. Since points $p \in M_f$ and $q \in M_g$ are only matched if $|f(p) - g(q)| < \gamma/4$, the matching is $(\gamma, \gamma/4)$-close. \hfill \qed

8 Discussions

Results on stability of topological persistence can be used in shape matching. If we take a dense point sample from the boundary of a shape, the distance functions to the shape boundary and and its point sample are similar. Therefore, if we have two similar shapes, the distance functions defined by their point samples are similar. As observed in previous works [1, 2], the results on persistence apply to such functions. Our results in this paper have some notable connections to a shape matching algorithm proposed by Dey et al. [4]. According to our results, we can expect that similar shapes have similar structures for maxima with large persistence in terms of the interval sets. The algorithm in [4] uses maxima and their stable manifolds for matching. We suspect that these stable manifolds are playing the role of connected components as suggested in this paper. Perhaps the performance of the matching algorithm in [4] now can be improved and better explained by our results. We plan to address this issue in future work.

Persistence diagrams [3] can be used to match shapes based on the critical values of the distance function. Does adding critical points increase the discrimination of this matching? Figure 6 provides such an example. The two unsimilar shapes in this figure has matching persistence diagrams. The distance function for each shape contains three local maxima and two saddle points. For each shape, two of the local maxima and one of the saddle points has long persistence while the other local maxima and saddle point have short persistence. (Persistence of saddle points which “create homology” groups is defined in [3].) Thus the persistence diagrams match.

Now assume that the two shapes are registered so that each $q_i$ lies on top of $p_i$. Even with such a registration, the local maxima $p_1$ does not match with $q_1$ since $p_1$ and $q_1$ have different persistence and $p_1$ does not match with $q_2$ or $q_3$ since they are not in the neighborhood of $p_1$ in the domain. Thus, our matching based on closeness of local maxima in both the range and domain distinguishes these shapes.

Figure 6: Two shapes with matching persistence diagram. Matching based on closeness in both the range and domain distinguishes these shapes. Local maxima $p_1, p_3, q_1, q_3$ and saddle points $p_5, q_6$ have long persistence. Local maxima $p_2, q_4$ and saddle points $p_4, q_4$ have short persistence.

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References


Appendix

Proof of Lemma 1. Assume function $f : \Omega \to \mathbb{R}$ is LR. Choose $\epsilon_1 > 0$ such that $f^{-1}(a)$ is a strong deformation retract of $\{x : a - \epsilon < f(x) < a + \epsilon\}$ for all $\epsilon < \epsilon_1$ and $a + \epsilon < b$. Similarly, choose $\epsilon_2 > 0$ such that $f^{-1}(b)$ is a strong deformation retract of $\{x : b - \epsilon < f(x) < b + \epsilon\}$ for all $\epsilon < \epsilon_2$ and $a < b - \epsilon$. Let $\epsilon_0$ be the minimum of $\epsilon_1$ and $\epsilon_2$ and $b - a$.

For any $\epsilon \leq \epsilon_0$, let $\phi_\epsilon^a$ be the mapping from $\{x : a - \epsilon < f(x) < a + \epsilon\} \times I$ to $f^{-1}(a)$ representing the strong deformation retract of $\left\{ a : a - \epsilon < x < a + \epsilon \right\}$ to $f^{-1}(a)$. Let $\phi_\epsilon^b$ be the mapping from $\{b : b - \epsilon < f(x) < b + \epsilon\} \times I$ to $f^{-1}(b)$ representing the strong deformation retract of $\left\{ b : b - \epsilon < x < b + \epsilon \right\}$ to $f^{-1}(b)$. Define

$$
\phi_\epsilon(x, t) = \begin{cases} 
\phi_\epsilon^a(x, t) & \text{for } a - \epsilon < x < a, \\
\phi_\epsilon^b(x, t) & \text{for } a \leq x \leq b, \\
x & \text{for } b < x < b + \epsilon.
\end{cases}
$$

$\phi_\epsilon$ is constant on $\{x : a \leq x \leq b\}$ and continuously deforms $\{x : a - \epsilon < x < a\}$ and $\{x : b - \epsilon < x < b + \epsilon\}$ onto $\{x : a \leq x \leq b\}$. Thus $\{x : a - \epsilon < f(x) < a\}$ is a strong deformation retract of $\{x : a - \epsilon < f(x) < a + \epsilon\}$ and $\{x : a - \epsilon < f(x) < a\}$ is a strong deformation retract of $\{x : a - \epsilon < f(x) < a + \epsilon\}$.

Proof of Lemma 2. Since $H(f^{-1}(a)) \to H(F_\infty^a)$ sends $h$ to zero, element $h$ is the boundary of a compact chain $C \subseteq F_\infty^a$. Chain $C$ is compact. (See [8, p. 711].) Thus $\{f(x) : x \in C\}$ is compact and has a maximum value $b'$. Since $C \subseteq F_\infty^b$, the mapping $H(f^{-1}(a)) \to H(F_\infty^b)$ sends $h$ to zero.

Let $b$ equal $\lim \inf \{ \beta : H(f^{-1}(a)) \to H(F_\infty^b) \}$ sends $h$ to zero. Note that $b \leq b'$. Since $f$ is LR, $H(f^{-1}(a))$ is isomorphic to $H(F_{\infty}^{a+b'}$) for sufficiently small $\epsilon$ and thus $H(f^{-1}(a)) \to H(F_{\infty}^{a+b'})$ does not send $h$ to zero. Thus $b$ is strictly greater than $a$.

Let $h'$ be the image of $h$ under the mapping $H(f^{-1}(a)) \to H(F_{\infty}^b)$. If $h'$ were zero, then $h'$ would be the boundary of some chain $C' \subseteq F_\infty^b$. Since $C'$ is compact, chain $C'$ would also be a subset of $F_\infty^b$ for some $b' < b$, contradicting the choice of $b$. Thus $h'$ is non-zero.

We show that $h'$ is destroyed by $f^{-1}(b)$. Since $f$ is LR, there is some $\epsilon_0 > 0$ such that $H(F_\infty^b \cup f^{-1}(b))$ is isomorphic to $H(F_{\infty}^{b+\epsilon})$ for all $\epsilon \leq \epsilon_0$. If $H(F_\infty^b) \to H(F_{\infty}^{b+\epsilon})$ does not map $h'$ to zero, then $H(F_\infty^b) \to H(F_{\infty}^{b+\epsilon})$ does not map $h'$ to zero for all $\epsilon \leq \epsilon_0$, and $b$ does not equal $\lim \inf \{ \beta : H(f^{-1}(b)) \to H(F_\infty^b) \}$ sends $h$ to zero. Thus $H(F_\infty^b) \to H(F_{\infty}^{b+\epsilon} \cup f^{-1}(b))$ maps $h'$ to zero and $h'$ is destroyed by $f^{-1}(b)$.

Proof of Lemma 7. Since $F_\infty^a$ is an oriented $d$-manifold and $p$ is a local maximum, some neighborhood $N_p$ of $p$ is homeomorphic to $\mathbb{R}^d$ and all points in $N_p \setminus \{p\}$ have value less than $f(p)$. Let $B$ be the unit ball in $\mathbb{R}^d$, and let $B_p$ be its image under the homeomorphism from $\mathbb{R}^d$ to $N_p$. Since all points in $N_p \setminus \{p\}$ have value less than $f(p)$, they are all in $F_\infty^a$.

By the Mayer-Vietoris Theorem, the sequence

$$H_k(B_p \setminus \{p\}) \to H_k(B_p) \oplus H_k(F_\infty^a \setminus \{p\}) \to H_k(F_\infty^a \cup \{p\})$$

is exact. Since the mapping $H_k(F_\infty^a) \to H_k(F_\infty^a \cup \{p\})$ sends $h$ to zero, the mapping $H_k(B_p) \oplus H_k(F_\infty^a \setminus \{p\}) \to H_k(F_\infty^a \cup \{p\})$ sends $(0 \oplus h)$ to zero. Since the sequence is exact, element $h$ is the image of some non-zero $h' \in H_k(B_p)$ under the mapping $H_k(B_p \setminus \{p\}) \to H_k(B_p) \oplus H_k(F_\infty^a \setminus \{p\})$. Since $H_k(B_p \setminus \{p\})$ is the zero group, for all $k \neq d - 1$, element $h'$ must be in $H_{d - 1}(B_p \setminus \{p\})$. Therefore, $h$ is an element of $H_{d - 1}(F_\infty^a)$ and so $k$ equals $d - 1$.

Lemma 10. Let $\Omega \subseteq \Omega'$ be topological spaces, let $\sigma_i, \ldots, \sigma_m'$ be the pathwise connected components of $\Omega'$, and let $\sigma$ equal $\sigma_i' \cap \Omega$ for each $i$. If the mapping $h(\Omega) \to h(\Omega')$ sends non-zero $h \in H_k(\Omega)$ to zero, then there exists some non-zero $h_i \in H_k(\sigma)$ such that the mapping $h(\sigma_i) \to h(\sigma_i' \cap \Omega)$ sends $h_i$ to zero. Moreover, if point $p$ is an element of $\Omega' - \Omega$, and the mapping $h(\sigma_i) \to h(\sigma_i' \setminus \{p\})$ does not send $h_i$ to zero, then $p$ is an element of $\sigma_i'$.

Proof. Since the $\sigma_i$ are pairwise disjoint, the homology group $H_k(\Omega)$ is isomorphic to $H_k(\sigma_1) \oplus \cdots \oplus H_k(\sigma_m')$. (See [8, Theorem 4.13, p. 69].) This isomorphism takes $h_i$ to $h_i(\Omega)$. Since $\sigma_i$ at least one of these $h_i$ must be non-zero.

Since the $\sigma_i'$ are pairwise disjoint, the homology group $H_k(\Omega')$ is isomorphic to $H_k(\sigma_i') \oplus \cdots \oplus H_k(\sigma_i_m')$. The mapping $H_k(\Omega) \to H_k(\Omega')$ sends $h_i$ to zero, so the mapping

$$H_k(\sigma_1) \oplus \cdots \oplus H_k(\sigma_m') \to H_k(\sigma_i') \oplus \cdots \oplus H_k(\sigma_i_m')$$

sends $(h_1(\sigma_1) \cdots h_m(\sigma_m))$ to $(0 \cdots 0)$. Thus, the mapping $H_k(\sigma_i) \to H_k(\sigma_i') \to H_k(\Omega')$, sends non-zero $h_i \in H_k(\sigma_i)$ to zero.

Let point $p$ be an element of $\Omega' - \Omega$ where the mapping $H_k(\sigma_i) \to H_k(\Omega' - \{p\})$ sends $h_i$ to some non-zero $h' \in H_k(\Omega' - \{p\})$. If $p \notin \sigma_i'$, then $\sigma_i' \subset \Omega' - \{p\}$ and the mapping $H_k(\sigma_i) \to H_k(\sigma_i' \setminus \{p\})$ sends $h_i$ to zero. Thus $\sigma_i'$ contains $p$.