# Complier Techniques for Efficient Parallelization of Out-of-Core Tensor Contractions*

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## Abstract

The Tensor Contraction Engine (TCE) is a domain-specific compiler for implementing complex tensor contraction expressions arising in quantum chemistry applications modeling electronic structure. This paper develops a performance model for tensor contractions, considering both disk I/O as well as inter-processor communication costs, to facilitate performance-model driven loop optimization for this domain. Experimental results are provided that demonstrate the accuracy and effectiveness of the model.

## 1 Introduction

The development of effective performance-model driven program transformation strategies for optimizing compilers is a challenging problem. We face this problem in the context of a domain-specific compiler targeted at a class of computationally demanding applications in quantum chemistry [2, 3]. A synthesis system is being developed for transformation into efficient parallel programs, of a high-level mathematical specification of a computation expressed as tensor contraction expressions. A tensor contraction is essentially a generalized matrix product involving multi-dimensional arrays. Often, the tensors are too large to fit in memory, so that out-of-core solutions are required. The optimization of a computation involving a collection of tensor contractions requires an accurate performance model for the core operation: a single tensor contraction, modeling both disk I/O costs and inter-processor communication costs. In this paper we address the problem of developing a performance model for parallel out-of-core tensor contractions.

The approach presented in this paper may be viewed as an example of the telescoping languages approach described in [14]. The telescoping languages/libraries approach aims at facilitating a high-level *scripting* interface for a domain-specific computation to the user, while achieving high performance that is portable across machine architectures, and compilation time that only grows linearly with the size of the user script. With this approach, library functions are pre-analyzed and appropriate annotations are included to provide information on performance characteristics. If user programs make heavy use of these library functions, the optimization of the user "script" is achieved using the performance characterization of the library functions, without requiring extensive analysis of the "expanded" program corresponding to inlined code for library functions. In a distributed computer, for efficient execution of out-of-core tensor contractions, two dominant overhead costs need to be reduced: inter-processor communication cost and local disk access cost. Many factors affect these costs, including the communication pattern, the parallel algorithms, data partitioning methods, loop orders, disk I/O placements and tile size selection. They are inter-related and can not be determined independently. The number of possible combinations is exceedingly large and searching them all is impractical. In this paper, we provide an approach, which can model the relationship between the space of possible structures and efficiently prune the search space to find the best solution in reasonable time.

This paper is organized as follows. In the next section, we introduce the main concepts and specify the parallel system supported by the algorithm. Section 3 discusses the impact of loop order and the placement of disk I/O statements. Algorithms used in *outside communication* pattern and *inside communication* pattern (defined in Section 2.1) are discussed in Section 4 and Section 5 respectively. Section 6 presents results from the application of the new algorithm to an example abstracted from NWChem [21]. We discuss related work in Section 7. Conclusions are provided in Section 8.

## 2 Preliminaries

Consider the following tensor contraction expression

$$C(a,b,c,d) = \sum_{m,n} A(a,b,m,n) \times B(c,d,m,n) \qquad (2.1)$$

where A and B are input arrays and C is the output array; $m,n$ are the summation indices. If all indices range over N, $O(N^6)$ arithmetic operations will be required to compute this.

Notice that a tensor contraction is essentially a generalized

*Supported in part by the National Science Foundation through the Information Technology Research program (CHE-0121676 and CHE-0121706), and NSF grants CCR-0073800 and EIA-9986052.

matrix multiplication. The above expression can written as

$$C(I,J) = A(I,K) \times B(J,K), \tag{2.2}$$

where $I \equiv \{a,b\}$, $J \equiv \{c,d\}$ and $K \equiv \{m,n\}$ are index sets considered as "macro-indices." This notation will be used in the rest of the paper. Consider a distributed-memory computer with $P$ processors in which every processor has limited local memory and unlimited local disk. If a processor needs data from the local disk of other processors, the required data will be first read by the owner processor and then communicated to the requesting processor. The inter-processor network bandwidth is denoted as $B_c$, and the local disk to memory bandwidth is $B_d$. Arrays A,B and C are either evenly distributed or fully replicated among all processors. An index set *dist* is used to represent the distribution pattern of an array. For example, if array A is distributed by index $i$ and $j$, then $A.dist$ is $\langle i,j \rangle$. If A is replicated on all processors, then $A.dist$ is $\langle \rangle$.

The original size of an array is denoted as *array.size*. In a parallel algorithm, the size of an array required for local computation is denoted as *array.localsize*. If all required data can fit into memory, there is no disk I/O involved. Otherwise the array whose *array.localsize* larger than one-third of local memory will be put on disk.

Because data sets are very large, we assume that both communication cost and disk I/O cost are dominated by the volume of data movement, not the number of data movements. The communication cost and disk I/O cost, can be calculated simply by dividing the transfered volume with the transferring bandwidth.

Three parallel matrix multiplication algorithms, *rotation, replication*, and *accumulation* are used. They will be discussed and compared in Section 2.2. The choice of the parallel algorithm decides the communication pattern. In the rotation algorithm, computation is executed in several steps. Each processor *circular-shifts* its local data with neighbors between computations. In the replication algorithm, one operand is *broadcasted* to all processors. In the accumulation algorithm, the partial result of the entire target array is *reduced* among all processors. These communication patterns can be implemented by corresponding communication routines. Communication routines on out-of-core data will be carried out in several steps and results in extra disk access cost.

## 2.1 Communication Methods

When there is insufficient memory to hold all the remote data for the next computation to be performed locally on a processor, we can use one of two broad approaches to handling the out-of-core data: 1) first perform disk-to-disk transfer so that all remote data needed by a processor for its portion of the computation is first moved to its local disk, or 2) intersperse disk I/O with computation on in-core sub-arrays. We call the first method as the **outside communication method**, and the latter as the **inside communication method**. In the **outside communication method**, communication and local computation are separated from each other. All remote data for the next computation is fetched before the start of the computation and stored on disk. It may cause redundant disk access, but achieves minimal communication cost. With the **inside communication method**, communication and local computation are interleaved together. When one block of data is copied into memory, the owner processor performs computations on it, and passes it to other processors requiring it. When other processors receive remote data, they perform computations on it, and discard it without writing it to disk. This approach incurs extra communication cost, but minimizes disk access cost. Examples of these two communication methods are shown in Figure 1 and Figure 2. The choice of the communication method introduces a trade-off between communication cost and disk access cost. Thus, when available local memory is large enough to hold all the remote data, we can directly select the outside communication method.

## 2.2 Parallel Algorithms and Distribution Indices

Many approaches have been proposed for implementing parallel matrix multiplication. In this framework, three simple and common parallel algorithms are considered as the basis for an individual tensor contraction: rotation, replication and accumulation. Implementation details of these parallel algorithms are discussed next.

1. **Rotation:** We use a generalization of Cannon's algorithm as the primary template. In this approach, a logical view of the $P$ processors as a two-dimensional $\sqrt{P} \times \sqrt{P}$ grid is used. To apply rotation parallel algorithm, each array is two-dimensional cyclic-block distributed along the two processor dimensions. A triplet $\{ i,j,k \}$ formed by one index from each index set I,J,and K defines a distribution $\langle i,j \rangle$ for the result array C, and distribution $\langle i,k \rangle$ and $\langle k,j \rangle$ for the input arrays A and B. The computation is carried out in $\sqrt{P}$ steps. One processor holds a sub-block of array A,B and C respectively at any moment, performs a sub-matrix multiplication on them and transfers blocks A and B to its neighbors after the computation is done.

2. **Replication:** In this scheme, each processor locally holds one full input array and a strip of the other two arrays. In order to achieve good performance, we always replicate the smaller operand. Without loss of generality, we assume the size of array A is less than the size of B. Thus, to use the replication parallel algorithm, array A is replicated on all processors, $A.dist = \langle \rangle$, and arrays B and C are distributed by the same dimensions belonging to the index set J, $B.dist = C.dist = \langle j \rangle, j \in \mathbf{J}$. Replication communication can be modeled as an all-to-all broadcast communication operation, whose communication cost is a topology-dependent function. To simplify the problem, we assume that the interconnection network is completely connected in the rest of the paper. Thus, we use the expression

$$Replicate(S) = (S.size)/B_c \tag{2.3}$$

to calculate the replication time.

3. **Accumulation:** In order to apply the accumulation parallel algorithm, two operands are distributed by the same summation indices, $A.dist = B.dist = \langle k \rangle, k \in \mathbf{K}$, and the target array is replicated on all processors, $C.dist = \langle \rangle$. Every processor executes a partial matrix multiplication and accumulates the result at last. The accumulation can be modeled as an all-to-all reduction communication operation, whose communication cost depend on the inter-

processor topology. In the completely-connected network, the all-to-all reduction cost is

$$Reduce(S) = (S.size \times log(P))/B_c \qquad (2.4)$$

If the distribution of the input or output arrays are not suitable for a specific parallel algorithm, we need to rearrange the data before or after executing the parallel algorithm. The redistribution procedure is separated from the computation procedure.

The pseudocode of these three parallel algorithms using the inside communication method are shown in Figure 1. The corresponding pseudocode for the outside communication method are shown in Figure 2. Arrays A, B and C are out-of-core arrays that are distributed using a block-cyclic distribution among *P* processors in order to render the *Collective* disk I/O operations load-balanced. *Collective* disk I/O operations operate on global tiles, which consist of a set of local tiles. The corresponding *local* disk I/O operation is indicated under the *collective* disk I/O operation. In the pseudocode, the loop order of the It, Jt and Kt loops is not determined, all the disk I/O statements and message passing statements are placed inside the innermost loop. However, after the loop structure is defined, these data movement statements will be inserted at the appropriate places in the actual program.

## 2.3 The Overall Problem Definition

Our overall goal is to develop a domain specific compiler, which can automatically translate a sequence of tensor expressions represented in high-level language to a high-performance parallel program in Fortran or C code. There are many methods to implement a parallel out-of-core tensor contraction. Different methods may perform differently because of differences in the hardware environment or because of the tensors' shape and size. In this section, we define the overall problem as following. For a given tensor contraction expression and some machine parameters, including the number of processors, the amount of physical available memory for every processor, the inter-processor network bandwidth, and the local disk to memory bandwidth, our goal is to determine:

- the communication method;
- the parallel algorithm and distributed indices;
- the order of the loops and disk I/O placements; and
- the tile sizes for each dimension

such that the total communication cost and the disk access cost are reduced.

For the input and output arrays, the algorithm can be used in either of these modes:

- the distribution of the input and output arrays are *unconstrained*, and can be chosen by the algorithm to optimize the communication cost; or
- the input and output arrays have a *constrained* distribution on to the processors in some pre-specified pattern.

The parallel execution can be decoupled into three stages:

1. redistribute the input arrays;
2. compute the tensor contraction expression in parallel; and
3. redistribute the output array.

| PA | Distribution Constraints. |
|---|---|
| Rotation | $A.dist = \langle i,k \rangle, B.dist = \langle j,k \rangle, C.dist = \langle i,j \rangle$ |
| Replication A | $B.dist = C.dist = \langle j \rangle$ |
| Replication B | $A.dist = C.dist = \langle i \rangle$ |
| Accumulation | $A.dist = B.dist = \langle k \rangle$ |

Table 1: Arrays distribution constraint for different parallel algorithms

The total execution time is the sum of execution times in the three stages. Because we only use load-balanced parallel algorithms, the computations are always evenly distributed among all the processors. We can ignore the calculation time, and consider only the communication overhead and the disk I/O overhead. The total overhead cost for a specific parallel algorithm, which is denoted as *PA*, can be calculated by:

$$\begin{aligned} Overhead(PA) &= Redist(A, A.dist1, A.dist2) \\ &+ Redist(B, B.dist1, B.dist2) \\ &+ Redist(C, C.dist1, C.dist2) \\ &+ Computation(A, B, C, PA), \end{aligned}$$

where A.dist1 and B.dist1 are the initial distribution of the input arrays A and B, and C.dist2 is the expected distribution of the output array C. A.dist2, and B.dist2 are operand distribution patterns required for *PA*. C.dist1 is the target distribution pattern generated by *PA*. A.dist2, B.dist2 and C.dist1 must be compatible with each other by the distribution constraints of *PA*. The distribution constraints for different parallel algorithms is shown in Table 1.

If the initial distribution is the same as the final distribution, data re-arrangement is not necessary, and the redistribution cost is zero. Otherwise, the redistribution cost is the sum of the communication cost and the disk I/O cost, which depend on the redistribution scheme and machine specific interprocessor topology.

When a parallel algorithm is chosen for matrix multiplication, suitable distribution methods of the input and output arrays are decided as well. However, in a multi-dimensional tensor contraction expression, many distribution methods can be applied in a specific parallel algorithm. The choice of the distribution method will affect the redistribution cost in stages one and three. However, the overhead of parallel execution in stage two can be calculated independently of the distribution method. Thus, in the following sections, we present an algorithm to determine all parameters, except for distribution method, which can minimize the overhead cost in stage two. The choice of the distribution method that allows for optimizing the redistribution cost will be discussed later.

# 3 Loop Order and Disk I/O Placements

In this section, we will concentrate on the loop order and the placements of disk I/O statements. We will consider only the order of tiling loops since different orders of the intra-tile loops will not significantly affect the execution time.

Consider the tensor contraction expression given in Expression ( 2.1). After tiling, the loops $It$, $Jt$, $Kt$ will be the tiling loops as shown in Figures 1 and 2. Note that $It$, $Jt$, $Kt$

```
for It,Jt,Kt
  Collective Read A_{Ii,Ki}
  (Local Read A_{Ii/√P,Ki/√P} )
  Collective Read B_{Ki,Ji}
  (Local Read B_{Ki/√P,Ji/√P} )
  Collective Read C_{Ii,Ji}
  (Local Read C_{Ii/√P,Ji/√P})
  for √P Rotations
    C_{Ii,Ji} += A_{Ii,Ki}*B_{Ki,Ji}
    Circular-shift in-core A_{Ii,Ki}
    Circular-shift in-core B_{Ki,Ji}
  Collective Write C_{Ii,Ji}
  (Local Write C_{Ii/√P,Ji/√P})
```

(a): Rotation

```
for It,Jt,Kt
  Collective Read A_{Ii,Ki}
  (Local Read A_{Ii,Ki}/P)
  A2A In-Core Broadcast A_{Ii,Ki}
  Collective Read B_{Ki,Ji}
  (Local Read B_{Ki,Ji/P})
  Collective Read C_{Ii,Ji}
  (Local Read C_{Ii,Ji/P})
  C_{Ii,Ji} += A_{Ii,Ki}*B_{Ki,Ji}
  Collective Write C_{Ii,Ji}
  (Local Write C_{Ii,Ji/P})
```

(b): Replication

```
for It,Jt,Kt
  Collective Read A_{Ii,Ki}
  (Local Read A_{Ii,Ki/P})
  Collective Read B_{Ki,Ji}
  (Local Read B_{Ki/P,Ji})
  Local Read C_{Ii,Ji}
  C_{Ii,Ji} += A_{Ii,Ki}*B_{Ki,Ji}
  All-Reduct In-Core C_{Ii,Ji}
  Local Write C_{Ii,Ji}
```

(c): Accumulation

Figure 1: Pseudocode of Inside Communication Method

```
for √P Rotations
  for It,Jt,Kt
    Collective Read A_{Ii,Ki}
    (Local Read A_{Ii/√P,Ki/√P} )
    Collective Read B_{Ki,Ji}
    (Local Read B_{Ki/√P,Ji/√P} )
    Collective Read C_{Ii,Ji}
    (Local Read C_{Ii/√P,Ji/√P})
    C_{Ii,Ji} += A_{Ii,Ki}*B_{Ki,Ji}
    Collective Write C_{Ii,Ji}
    (Local Write C_{Ii/√P,Ji/√P})
  Circular-shift Out-of-Core A_{I,K}
  Circular-shift Out-of-Core B_{K,J}
```

(a): Rotation

```
A2A Broadcast Out-of-Core A_{I,K}
for It,Jt,Kt
  Local Read A_{Ii,Ki}
  Collective Read B_{Ki,Ji}
  (Local Read B_{Ki,Ji/P})
  Collective Read C_{Ii,Ji}
  (Local Read C_{Ii,Ji/P})
  C_{Ii,Ji} += A_{Ii,Ki}*B_{Ki,Ji}
  Collective Write C_{Ii,Ji}
  (Local Write C_{Ii,Ji/P})
```

(b): Replication

```
for It,Jt,Kt
  Collective Read A_{Ii,Ki}
  (Local Read A_{Ii,Ki/P})
  Collective Read B_{Ki,Ji}
  (Local Read B_{Ki/P,Ji})
  Local Read C_{Ii,Ji}
  C_{Ii,Ji} += A_{Ii,Ki}*B_{Ki,Ji}
  Local Write C_{Ii,Ji}
All-Reduct Out-of-Core C_{I,J}
```

(c): Accumulation

Figure 2: Pseudocode of Outside Communication Method

are not single indices, but index sets, i.e., they each consist of several loop indices or be empty. Orderings of these tiling loops will depend upon the order of the placement of the disk I/O statements. There are three disk read statements corresponding to the three arrays $A$, $B$, and $C$. We need to consider six cases for the placement of read statements: *ABC*, *ACB*, *BAC*, *BCA*, *CAB*, *CBA*.

Consider the case where read statements are in the order *ABC* as shown in Figure 3. The three read statements will divide the tiling loops into four parts: $D_1$, $D_2$, $D_3$, and $D_4$. Each of these parts will contain some loops from each of the index sets $It$, $Jt$, $Kt$. Let $D_i$ contain index sets $It_i$, $Jt_i$, $Kt_i$ as shown in Figure 3(a). Considering the loops in part $D_1$, we note that if $Jt_1$ is non-empty, then disk I/O for $A$ will be unnecessarily repeated several times. So $Jt_1$ should be moved to part $D_2$ to reduce the total volume of disk access for $A$ without increasing the size of local buffers for $A$, $B$ and $C$. After putting $Jt_1$ to part $D_2$, we can merge index sets $Jt_1$ and $Jt_2$ together, and re-name the new index set as $Jt_1$. Considering the loops in part $D_2$, we note that if $It_2$ is non-empty, then disk I/O for $B$ will be unnecessarily repeated several times. So $It_2$ should be moved to part $D_3$ to reduce the total volume of disk access for $B$ without increaseing the size of local buffers. Further, $Kt_2$ would be empty or moved to part $D_1$ to reduce the memory requirement for the local buffer of $A$ without increasing the volume of disk access for $A$, $B$ and $C$. Similarly, considering the loops in part $D_3$, we note that $Kt_3$ should be empty or be moved to part $D_4$ to reduce the total volume of disk access for $C$ and that the loops in $Jt_3$ should be empty or moved to part $D_2$ to reduce the memory requirement for disk access of $B$. Continuing in this fashion, we decide to put loops in $It_4$ in part $D_3$ and loops in $Jt_4$ in part $D_2$.

The simplified code is shown in Figure 3(b). Note, that the particular loops put in index sets will not affect the minimum Overhead cost, but they will determine whether the conditions under which we can achieve the minimum Overhead cost are satisfied or not. This will be explained in detail in later sections.

## 4 Overhead Minimization for the Outside Communication Method

In this section, we analyze each of the three parallel algorithms (rotation, replication and accumulation) with the **outside communication** pattern and determine the minimal *Overhead* cost achievable along with the conditions under which this will be possible. In the expressions used in this and the next section, $A, B, C$ will denote the sizes of arrays $A, B, C$, respectively; the terms $I$, $J$, $K$ and $It_1$, $Jt_1$, $Kt_1$ will denote the corresponding loop bounds. The total number of processors will be denoted by $P$ and the local memory available for the tiles of each array, which we assume to be one-third of the local memory per processor, is denoted by $M$. The combined memory of all processors is, therefore, $M \times P$.

### 4.1 Rotation

Let us consider the tensor contraction code with disk I/O placement order *ABC*, the outside communication pattern, and

```
for It1, Jt1, Kt1
  Read A_{Ii1,Ki1,I2,I3,I4,K2,K3,K4}
  for It2, Jt2, Kt2
    Read B_{Ji1,Ji2,Ki1,Ki2,J3,J4,K3,K4}
    for It3, Jt3, Kt3
      Read C_{Ii1,Ii2,Ii3,I4,Ji1,Ji2,Ji3,J4}
      for It4, Jt4, Kt4
        [ C+ = A × B
```

(a) Initial groups

```
for It1, Kt1
  Read A_{Ii1,Ki1,I2,K2}
  for Jt1
    Read B_{Ji1,Ki1,K2}
    for It2
      Read C_{Ii1,Ii2,Ki1,Ki2}
      for Kt2
        [ C+ = A × B
```

(b) After cleanup

Figure 3: Loop groups

rotation type of parallelism as shown in Figure 2(a). The tiling loops are ordered as discussed in the previous section. Our goal is to determine the tile sizes (or the number of tiles) that will minimize the *Overhead* cost, including disk I/O cost and communication cost.

In this case, each of the three arrays are partitioned equally among the $P$ processors. So we have $A.localsize = A/P$, $B.localsize = B/P$, and $C.localsize = C/P$. The communication corresponds to shifting the $A$ and $B$ arrays to adjacent processors. These communications happen $\sqrt{P}$ times and each of these also involves disk operations. Therefore, the total communication volume $\mathcal{V} = \sqrt{P} \times (\frac{A}{P} + \frac{B}{P}) = (\frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}})$. The disk access volume during communication $\mathcal{D}_1 = 2 \times (\frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}})$, since the disk is accessed twice, once for reading and once for writing. It is clear that these two terms are independent of the tile sizes. The disk access volume during the computation $\mathcal{D}_2 = \sqrt{P} \times (\frac{A}{P} + \frac{B}{P} \times It_1 + 2 \times \frac{C}{P} \times Kt_1) = \frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}} \times It_1 + 2 \times \frac{C}{\sqrt{P}} \times Kt_1$. The total disk access volume $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$. For simplicity, in the calculations below $\mathcal{D}$ will include only those two parts that depend on the number of tiles (or tile sizes).

It is clear that this term depends on the number of tiles. To minimize Overhead cost, we will have to minimize the disk access volume during the computation and hence $It_1, Kt_1$ should be made 1. But this is not possible due to the constraint that the tiles of array $A$, $B$ and $C$ should fit into memory. Here we assume that each of these array tiles occupies one third of the memory. The constraints involving tiles can be expressed as follows.

$$It_1 \times Kt_1 \geq \frac{A}{M \times P} \qquad (4.1)$$

$$Jt_1 \times Kt_1 \geq \frac{B}{M \times P} \qquad (4.2)$$

$$It_1 \times It_2 \times Jt_1 \geq \frac{C}{M \times P} \qquad (4.3)$$

Note that only Eqn. 4.1 involves both $It_1$ and $Kt_1$, which we want to be 1. We will try to minimize $\mathcal{D}$ under the constraint of Eqn. 4.1. The other two equations can be simultaneously satisfied by using a large value the the unconstrained variables $It_2$ and $Jt_1$. Since we are trying to reduce the values of $It_1$ and $Kt_1$ while satisfying Eqn. 4.1, the Eqn. 4.1 can be written as $It_1 \times Kt_1 = \frac{A}{M \times P}$. With this modification, we can substitute the value of $Kt_1$ in the equation for $\mathcal{D}$ to get,

$$B \times It_1^2 - \sqrt{P} \times \mathcal{D} \times It_1 + \frac{2 \times C \times A}{M \times P} = 0 \qquad (4.4)$$

The above quadratic equation will have a real solution under the condition that the quadratic curve discriminant $P \times \mathcal{D}^2 - 4 \times B \times (\frac{2 \times C \times A}{M \times P}) \geq 0$. In other words, for any real value of $It_1$, the minimum achievable value of $\mathcal{D}$ is $\frac{1}{P}\sqrt{\frac{8 \times A \times B \times C}{M}}$. This minimum value of $\mathcal{D}$ can be achieved with $It_1 = I \times \sqrt{\frac{2}{M \times P}}$ and $Kt_1 = \frac{K}{\sqrt{2 \times M \times P}}$. In order to satisfy Equations 4.2 and 4.3, we need to choose values of $Jt_1$ and $It_2$ that satisfy the conditions $Jt_1 \geq J \times \sqrt{\frac{2}{M \times P}}$ and $It_2 \geq 1$. Hence, the minimum total disk access volume is

$$2 \times (\frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}}) + \frac{A}{\sqrt{P}} + \frac{1}{P}\sqrt{\frac{8 \times A \times B \times C}{M}}. \qquad (4.5)$$

If these values are not integers, the number of tiles will be set to the ceiling. There are two special cases if values of $It_1$ or $Kt_1$ are less than 1.

- *Case 1:* $I < \sqrt{\frac{M \times P}{2}}$

  In this case, we select the values as $It_1 = 1$, $Kt_1 = \frac{A}{M \times P}$, $Jt_1 \geq \frac{J}{I}$, $It_2 \geq \frac{I^2}{M \times P}$. The minimum total disk access volume during the computation in this case will be $\frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}} + 2 \times \frac{C}{\sqrt{P}} \times \frac{A}{M \times P}$.

- *Case 2:* $K < \sqrt{2 \times M \times P}$

  In this case, we select the values as $It_1 = \frac{A}{M \times P}$, $Kt_1 = 1$, $Jt_1 \geq \frac{B}{M \times P}$, $It_2 \geq \frac{M \times P}{K^2}$. The minimum total disk access volume during the computation in this case will be $\frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}} \times \frac{A}{M \times P} + 2 \times \frac{C}{\sqrt{P}}$.

We performed the analysis for the other five disk placement orders in a similar fashion. The results are shown in Table 2.

## 4.2 Replication

For this case, let us consider the tensor contraction code with disk I/O placement order *ABC*, outside communication pattern, and replication type of parallelism as shown in Figure 2(b). The tiling loops are ordered as discussed in the previous section. As in the case of rotation, our goal is to determine the tile sizes to minimize the Overhead cost.

Without loss of generality, we assume array $A$ is smaller than array $B$. Thus, the arrays $B$ and $C$ are partitioned equally among the $P$ processors whereas $A$ is replicated on all processors. So we have $A.localsize = A$, $B.localsize = B/P$, and $C.localsize = C/P$. In this case, communication corresponds to broadcasting array $A$. Therefore, the total communication volume $\mathcal{V} = A$. The disk access volume during communication $\mathcal{D}_1 = A$. Also in this case the above two terms are independent of the tile sizes. The disk access volume during the computation $\mathcal{D}_2 = A + \frac{B}{P} \times It_1 + 2 \times \frac{C}{P} \times Kt_1$. The total disk access volume $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$.

It is clear that D depends on the number of tiles. To minimize the Overhead cost, we will have to minimize the disk access volume during the computation and hence $It_1$, $Kt_1$ should be set to 1. But this is not possible due to the constraint that the tiles of arrays $A$, $B$, and $C$ fit into memory. The size constraints involving tiles can be expressed as follows.

$$It_1 \times Kt_1 \geq \frac{A}{M} \quad (4.6)$$

$$Jt_1 \times Kt_1 \geq \frac{B}{M \times P} \quad (4.7)$$

$$It_1 \times It_2 \times Jt_1 \geq \frac{C}{M \times P} \quad (4.8)$$

Our analysis here is similar to that for the case of rotation (Section 4.1). We will try to minimize $\mathcal{D}$ under the constraint of Eqn. 4.6. The other two equations can be simultaneously satisfied by using a large value for the unconstrained variables $It_2$ and $Jt_1$. Since we are trying to reduce the values of $It_1$ and $Kt_1$ while satisfying Eqn. 4.6, the Eqn. 4.6 can be written as $It_1 \times Kt_1 = \frac{A}{M}$. With this modification, we can substitute the value of $Kt_1$ in the equation for $\mathcal{D}$ to get

$$B \times It_1^2 - P \times \mathcal{D} \times It_1 + \frac{2 \times C \times A}{M} = 0. \quad (4.9)$$

From the above equation, it should be clear that for any real value of $It_1$, the minimum achievable value of $\mathcal{D}$ is $\frac{1}{P}\sqrt{\frac{8 \times A \times B \times C}{M}}$. This minimum value of $\mathcal{D}$ can be achieved with $It_1 = I \times \sqrt{\frac{2}{M}}$ and $Kt_1 = \frac{K}{\sqrt{2 \times M}}$, $Jt_1 \geq \frac{J}{P} \times \sqrt{\frac{2}{M}}$ and $It_2 \geq 1$. These values satisfy all the constraints. Hence, the minimum total disk access volume is

$$A + A + \frac{1}{P}\sqrt{\frac{8 \times A \times B \times C}{M}} \quad (4.10)$$

If these values are not integers, the number of tiles will be set to the ceiling. There are two special cases if the values of $It_1$ or $Kt_1$ are less than 1. The analysis for these cases can be done as shown in Section 4.1.

We did the analysis for the other five disk placement orders as above. The results are shown in Table 3.

### 4.3 Accumulation

In this section, we deal with the accumulation type of parallelism. Consider the tensor contraction code with accumulation type of parallelism as shown in Figure 2(c). Again our goal is to determine the tile sizes that will minimize the total Overhead cost.

In this case, arrays $A$ and $B$ are partitioned equally among the $P$ processors whereas $C$ is replicated on all processors. So we have $A.localsize = A/P$, $B.localsize = B/P$, $C.localsize = C$. In this case, the communication involves an All-Reduce operation of array $C$. Therefore, total communication volume $\mathcal{V} = C \times \log P$. The disk access volume during communication $\mathcal{D}_1 = C$. Again the total communication cost is independent of the tile sizes. The disk access volume during the computation $\mathcal{D}_2 = \frac{A}{P} + \frac{B}{P} \times It_1 + 2 \times C \times Kt_1$. The total disk access volume $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$.

As in the previous sections, to minimize the disk access volume during the computation, $It_1$ and $Kt_1$ should be made 1. But this is prevented by the constraint that the tiles of array $A$, $B$, and $C$ should fit into memory. The constraints involving tiles in this case can be expressed as follows.

$$It_1 \times Kt_1 \geq \frac{A}{M \times P} \quad (4.11)$$

$$Jt_1 \times Kt_1 \geq \frac{B}{M \times P} \quad (4.12)$$

$$It_1 \times It_2 \times Jt_1 \geq \frac{C}{M} \quad (4.13)$$

We do the analysis similar to that in the previous subsections. We will try to minimize $\mathcal{D}$ under the constraint of Eqn. 4.11. The other two equations are simultaneously satisfied by using a large value for the unconstrained variables $It_2$ and $Jt_1$. As before, the Eqn. 4.11 can be written as $It_1 \times Kt_1 = \frac{A}{M \times P}$. Now substituting the value of $Kt_1$ in the equation for $\mathcal{D}$ we get

$$\frac{B}{P} \times It_1^2 - \mathcal{D} \times It_1 + \frac{2 \times C \times A}{M \times P} = 0 \quad (4.14)$$

From this equation it is clear that, for any real value of $It_1$, the minimum achievable value of $\mathcal{D}$ is $\frac{1}{P}\sqrt{\frac{8 \times A \times B \times C}{M}}$. This minimum value of $\mathcal{D}$ can be achieved with $It_1 = I \times \sqrt{\frac{2}{M}}$, $Kt_1 = \frac{K}{P \times \sqrt{2 \times M}}$, $Jt_1 \geq J \times \sqrt{\frac{2}{M}}$, and $It_2 \geq 1$. Hence, the minimum total disk access volume is

$$C + \frac{A}{P} + \frac{1}{P}\sqrt{\frac{8 \times A \times B \times C}{M}} \quad (4.15)$$

If these values are not integers, the number of tiles will be set to the ceiling. There are two special cases if the values of $It_1$ or $Kt_1$ are less than 1. Again, the analysis for these cases can be done as shown in the previous subsections.

We did the analysis for the other five disk placement orders as above. The results are shown in Table 4.

## 5 Overhead Minimization for the Inside Communication Method

In this section, we analyze each of the three parallel algorithms possible with the **inside communication** pattern and determine the minimal Overhead cost achievable along with the conditions under which this will be possible.

### 5.1 Rotation

Consider the tensor contraction code with disk I/O placement order *ABC*, inside communication pattern, and the rotation type of parallelism as shown in Figure 1(a). The tiling loop ordering is decided as before. The goal is to determine the tile sizes (or the number of tiles) that will minimize the total Overhead cost.

In this case, each of the three arrays are partitioned equally among the $P$ processors in a block-cyclic fashion. So we have $A.localsize = A/P$, $B.localsize = B/P$, $C.localsize = C/P$. The communication corresponds to shifting the $A$ and $B$ arrays to adjacent processors. This communication happens $\sqrt{P}$ times for each iteration of the tiling loops and each of these also

| Disk Order | Cost Estimate Formulas | Tile Constraints | Minimal Disk Cost and conditions |
| --- | --- | --- | --- |
| ABC/<br>ACB | $\mathcal{V} = \frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}}$<br>$\mathcal{D}_1 = \frac{2}{\sqrt{P}}(A+B)$<br>$\mathcal{D}_2 = \sqrt{P} \times (\frac{A}{P} + \frac{B}{P} \times It1 + 2 \times \frac{C}{P} \times Kt1)$ | $It1 \times Kt1 \geq \frac{A}{M \times P}$<br>$Jt1 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times It2 \times Jt1 \geq \frac{C}{M \times P}$ | $< 1 >$ If $I \geq \sqrt{\frac{M \times P}{2}}$, and $K \geq \sqrt{2M \times P}$, then<br>$It1 = I\sqrt{\frac{2}{M \times P}}$, $Kt1 = \frac{K}{\sqrt{2M \times P}}$, $Jt1 \geq J\sqrt{\frac{2}{M \times P}}$, $It2 = 1$<br>$\mathcal{D}_2 = \frac{A}{\sqrt{P}} + \frac{\sqrt{8ABC}}{P\sqrt{M}}$<br><br>$< 2 >$ If $I < \sqrt{\frac{M \times P}{2}}$, and $K \geq \sqrt{2M \times P}$, then<br>$It1 = 1$, $Kt1 = \frac{A}{M \times P}$, $Jt1 \geq \frac{J}{I}$, $It2 = 1$<br>$\mathcal{D}_2 = \frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}} + \frac{2AC}{MP\sqrt{P}}$<br><br>$< 3 >$ If $I \geq \sqrt{\frac{M \times P}{2}}$, and $K < \sqrt{2M \times P}$, then<br>$It1 = \frac{A}{M \times P}$, $Kt1 = 1$, $Jt1 \geq \frac{B}{M \times P}$, $It2 = 1$<br>$\mathcal{D}_2 = \frac{A}{\sqrt{P}} + \frac{A \times B}{MP\sqrt{P}} + \frac{2C}{\sqrt{P}}$ |
| BAC/<br>BAC | $\mathcal{V} = \frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}}$<br>$\mathcal{D}_1 = \frac{2}{\sqrt{P}}(A+B)$<br>$\mathcal{D}_2 = \sqrt{P} \times (\frac{A}{P} \times Jt1 + \frac{B}{P} + 2 \times \frac{C}{P} \times Kt1)$ | $It1 \times Kt1 \geq \frac{A}{M \times P}$<br>$Jt1 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times Jt1 \times Jt2 \geq \frac{C}{M \times P}$ | $< 1 >$ If $J \geq \sqrt{\frac{M \times P}{2}}$, and $K \geq \sqrt{2M \times P}$, then<br>$It1 \geq I\sqrt{\frac{2}{M \times P}}$, $Kt1 = \frac{K}{\sqrt{2M \times P}}$, $Jt1 = J\sqrt{\frac{2}{M \times P}}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{B}{\sqrt{P}} + \frac{\sqrt{8ABC}}{P\sqrt{M}}$<br><br>$< 2 >$ If $J < \sqrt{\frac{M \times P}{2}}$, and $K \geq \sqrt{2M \times P}$, then<br>$Jt1 = 1$, $Kt1 = \frac{B}{M \times P}$, $It1 \geq \frac{I}{J}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}} + \frac{2BC}{MP\sqrt{P}}$<br><br>$< 3 >$ If $J \geq \sqrt{\frac{M \times P}{2}}$, and $K < \sqrt{2M \times P}$, then<br>$Jt1 = \frac{B}{M \times P}$, $Kt1 = 1$, $It1 \geq \frac{A}{M \times P}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{B}{\sqrt{P}} + \frac{A \times B}{MP\sqrt{P}} + \frac{2C}{\sqrt{P}}$ |
| CAB/<br>CBA | $\mathcal{V} = \frac{A}{\sqrt{P}} + \frac{B}{\sqrt{P}}$<br>$\mathcal{D}_1 = \frac{2}{\sqrt{P}}(A+B)$<br>$\mathcal{D}_2 = \sqrt{P} \times (\frac{A}{P} \times Jt1 + \frac{B}{P} \times It1 + 2 \times \frac{C}{P})$ | $It1 \times Kt1 \geq \frac{A}{M \times P}$<br>$Jt1 \times Jt2 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times Jt1 \geq \frac{C}{M \times P}$ | $< 1 >$ If $I \geq \sqrt{M}$, and $J \geq \sqrt{M}$, then<br>$It1 = \frac{I}{\sqrt{M}}$, $Kt1 \geq \frac{K}{\sqrt{M}}$, $Jt1 = \frac{J}{\sqrt{M}}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{2C}{\sqrt{P}} + \frac{\sqrt{4ABC}}{P\sqrt{M}}$<br><br>$< 2 >$ If $I < \sqrt{M}$, and $J \geq \sqrt{M}$, then<br>$It1 = 1$, $Jt1 = \frac{C}{M \times P}$, $Kt1 \geq \frac{A}{M \times P}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{AC}{MP\sqrt{P}} + \frac{B}{\sqrt{P}} + \frac{2C}{\sqrt{P}}$<br><br>$< 3 >$ If $I \geq \sqrt{M}$, and $J < \sqrt{M}$, then<br>$It1 = \frac{C}{M \times P}$, $Jt1 = 1$, $Kt1 \geq \frac{B}{M \times P}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{A}{\sqrt{P}} + \frac{BC}{MP\sqrt{P}} + \frac{2C}{\sqrt{P}}$ |

Table 2: Communication and Disk Access Volume for the **Outside/Rotation** pattern

| Disk Order | Cost Estimate Formulas | Tile Constraints | Minimal Disk Cost and conditions |
|---|---|---|---|
| ABC/<br>ACB | $\mathcal{V} = A$<br>$\mathcal{D}_1 = A$<br>$\mathcal{D}_2 = A + \frac{B}{P} \times It1 + 2 \times \frac{C}{P} \times Kt1$ | $It1 \times Kt1 \geq \frac{A}{M}$<br>$Jt1 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times It2 \times Jt1 \geq \frac{C}{M \times P}$ | $<1>$ If $I \geq \sqrt{\frac{M}{2}}$, and $K \geq \sqrt{2M}$, then<br>$It1 = I\sqrt{\frac{2}{M}}, Kt1 = \frac{K}{\sqrt{2M}}, Jt1 \geq J\frac{\sqrt{2}}{\sqrt{MP}}, It2 = 1$<br>$\mathcal{D}_2 = A + \frac{\sqrt{8ABC}}{\sqrt{MP}}$<br><br>$<2>$ If $I < \sqrt{\frac{M}{2}}$, and $K \geq \sqrt{2M}$, then<br>$It1 = 1, Kt1 = \frac{A}{M}, Jt1 \geq \frac{J}{I \times P}, It2 = 1$<br>$\mathcal{D}_2 = A + \frac{B}{P} + \frac{2AC}{MP}$<br><br>$<3>$ If $I \geq \sqrt{\frac{M}{2}}$, and $K < \sqrt{2M}$, then<br>$It1 = \frac{A}{M}, Kt1 = 1, Jt1 \geq \frac{B}{M \times P}, It2 = 1$<br>$\mathcal{D}_2 = A + \frac{AB}{MP} + \frac{2C}{P}$ |
| BAC/<br>BCA | $\mathcal{V} = A$<br>$\mathcal{D}_1 = A$<br>$\mathcal{D}_2 = A \times Jt1 + \frac{B}{P} + 2 \times \frac{C}{P} \times Kt1$ | $It1 \times Kt1 \geq \frac{A}{M}$<br>$Jt1 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times Jt1 \times Jt2 \geq \frac{C}{M \times P}$ | $<1>$ If $J \geq \frac{\sqrt{M} \times P}{\sqrt{2}}$, and $K \geq \sqrt{2M}$, then<br>$It1 \geq I\sqrt{\frac{2}{M}}, Kt1 = \frac{K}{\sqrt{2M}}, Jt1 = J\sqrt{\frac{\sqrt{2}}{\sqrt{MP}}}, Jt2 = 1$<br>$\mathcal{D}_2 = \frac{B}{P} + \frac{\sqrt{8ABC}}{P\sqrt{M}}$<br><br>$<2>$ If $J < \frac{\sqrt{M} \times P}{\sqrt{2}}$, and $K \geq \sqrt{2M}$, then<br>$Jt1 = 1, Kt1 = \frac{B}{M \times P}, It1 \geq \frac{IP}{J}, Jt2 = 1$<br>$\mathcal{D}_2 = A + \frac{B}{P} + \frac{2BC}{MP}$<br><br>$<3>$ If $J \geq \frac{\sqrt{M} \times P}{\sqrt{2}}$, and $K < \sqrt{2M}$, then<br>$Jt1 = \frac{B}{M \times P}, Kt1 = 1, It1 \geq \frac{A}{M}, Jt2 = 1$<br>$\mathcal{D}_2 = \frac{A \times B}{MP} + \frac{B}{P} + \frac{2C}{P}$ |
| CAB/<br>CBA | $\mathcal{V} = A$<br>$\mathcal{D}_1 = A$<br>$\mathcal{D}_2 = A \times Jt1 + \frac{B}{P} \times It1 + 2 \times \frac{C}{P}$ | $It1 \times Kt1 \geq \frac{A}{M}$<br>$Jt1 \times Jt2 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times Jt1 \geq \frac{C}{M \times P}$ | $<1>$ If $I \geq \sqrt{M}$, and $J \geq \sqrt{M} \times P$, then<br>$It1 = \frac{I}{\sqrt{M}}, Kt1 \geq \frac{K}{\sqrt{M}}, Jt1 = \frac{J}{\sqrt{MP}}, Jt2 = 1$<br>$\mathcal{D}_2 = \frac{2C}{P} + \frac{\sqrt{4ABC}}{P\sqrt{M}}$<br><br>$<2>$ If $I < \sqrt{M}$, and $J \geq \sqrt{M} \times P$, then<br>$It1 = 1, Jt1 = \frac{C}{M \times P}, Kt1 \geq \frac{A}{M}, Jt2 = 1$<br>$\mathcal{D}_2 = \frac{AC}{MP} + \frac{B}{P} + \frac{2C}{P}$<br><br>$<3>$ If $I \geq \sqrt{M}$, and $J < \sqrt{M} \times P$, then<br>$It1 = \frac{C}{M \times P}, Jt1 = 1, Kt1 \geq \frac{KP}{J}, Jt2 = 1$<br>$\mathcal{D}_2 = A + \frac{BC}{MP} + \frac{2C}{P}$ |

Table 3: Communication and Disk Access Volume for the **Outside/Replication** pattern

| Disk Order | Cost Estimate Formulas | Tile Constraints | Minimal Disk Cost and conditions |
|---|---|---|---|
| ABC/<br>ACB | $\mathcal{V} = C \times logP$<br>$\mathcal{D}_1 = C$<br>$\mathcal{D}_2 = \frac{A}{P} + \frac{B}{P} \times It1 + 2C \times Kt1$ | $It1 \times Kt1 \geq \frac{A}{M \times P}$<br>$Jt1 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times It2 \times Jt1 \geq \frac{C}{M}$ | $<1>$ If $I \geq \sqrt{\frac{M}{2}}$, and $K \geq \sqrt{2M} \times P$, then<br>$It1 = I\sqrt{\frac{2}{M}}$, $Kt1 = \frac{K}{\sqrt{2MP}}$, $Jt1 \geq J\sqrt{\frac{2}{M}}$, $It2 = 1$<br>$\mathcal{D}_2 = \frac{A}{P} + \frac{\sqrt{8ABC}}{\sqrt{MP}}$<br><br>$<2>$ If $I < \sqrt{\frac{M}{2}}$, and $K \geq \sqrt{2M} \times P$, then<br>$It1 = 1$, $Kt1 = \frac{A}{MP}$, $Jt1 \geq \frac{J}{I}$, $It2 = 1$<br>$\mathcal{D}_2 = \frac{A}{P} + \frac{B}{P} + \frac{2AC}{MP}$<br><br>$<3>$ If $I \geq \sqrt{\frac{M}{2}}$, and $K < \sqrt{2M} \times P$, then<br>$It1 = \frac{A}{MP}$, $Kt1 = 1$, $Jt1 \geq \frac{B}{M \times P}$, $It2 = 1$<br>$\mathcal{D}_2 = A + \frac{AB}{M \times P^2} + 2C$ |
| BAC/<br>BCA | $\mathcal{V} = C \times logP$<br>$\mathcal{D}_1 = C$<br>$\mathcal{D}_2 = \frac{A}{P} \times Jt1 + \frac{B}{P} + 2 \times \frac{C}{P} \times Kt1$ | $It1 \times Kt1 \geq \frac{A}{M \times P}$<br>$Jt1 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times Jt1 \times Jt2 \geq \frac{C}{M}$ | $<1>$ If $J \geq \sqrt{\frac{M}{2}}$, and $K \geq \sqrt{2M} \times P$, then<br>$It1 \geq I\sqrt{\frac{2}{M}}$, $Kt1 = \frac{K}{\sqrt{2MP}}$, $Jt1 = J\sqrt{\frac{\sqrt{2}}{\sqrt{M}}}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{B}{P} + \frac{\sqrt{8ABC}}{P\sqrt{M}}$<br><br>$<2>$ If $J < \sqrt{\frac{M}{2}}$, and $K \geq \sqrt{2M} \times P$, then<br>$Jt1 = 1$, $Kt1 = \frac{B}{M \times P}$, $It1 \geq \frac{C}{M}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{A}{P} + \frac{B}{P} + \frac{2BC}{MP}$<br><br>$<3>$ If $J \geq \sqrt{\frac{M}{2}}$, and $K < \sqrt{2M} \times P$, then<br>$Jt1 = \frac{B}{M \times P}$, $Kt1 = 1$, $It1 \geq \frac{A}{M}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{AB}{MP} + \frac{B}{P} + 2C$ |
| CAB/<br>CBA | $\mathcal{V} = C \times LogP$<br>$\mathcal{D}_1 = C$<br>$\mathcal{D}_2 = \frac{A}{P} \times Jt1 + \frac{B}{P} \times It1 + 2C$ | $It1 \times Kt1 \geq \frac{A}{M \times P}$<br>$Jt1 \times Jt2 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times Jt1 \geq \frac{C}{M}$ | $<1>$ If $I \geq \sqrt{M}$, and $J \geq \sqrt{M}$, then<br>$It1 = \frac{I}{\sqrt{M}}$, $Kt1 \geq \frac{K}{\sqrt{MP}}$, $Jt1 = \frac{J}{\sqrt{M}}$, $Jt2 = 1$<br>$\mathcal{D}_2 = 2C + \frac{\sqrt{4ABC}}{P\sqrt{M}}$<br><br>$<2>$ If $I < \sqrt{M}$, and $J \geq \sqrt{M}$, then<br>$It1 = 1$, $Jt1 = \frac{C}{M}$, $Kt1 \geq \frac{A}{MP}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{AC}{MP} + \frac{B}{P} + 2C$<br><br>$<3>$ If $I \geq \sqrt{M}$, and $J < \sqrt{M}$, then<br>$It1 = \frac{C}{M}$, $Jt1 = 1$, $Kt1 \geq \frac{B}{PM}$, $Jt2 = 1$<br>$\mathcal{D}_2 = \frac{A}{P} + \frac{BC}{MP} + 2C$ |

Table 4: Communication and Disk Access Volume for the **Outside/Accumulation** pattern

| | $I \geq \sqrt{MP}, J \geq \sqrt{MP}$, $K \geq \sqrt{MP}$ | $I < \sqrt{MP}, J \geq \sqrt{MP}$, $K \geq \sqrt{MP}$ | $I \geq \sqrt{MP}, J < \sqrt{MP}$, $K \geq \sqrt{MP}$ | $I \geq \sqrt{MP}, J \geq \sqrt{MP}$, $K < \sqrt{MP}$ |
|---|---|---|---|---|
| *ABC* | $\mathcal{D} = \frac{A}{P} + 3\sqrt{\frac{ABC}{MP^3}}$ $\mathcal{V} = 2\sqrt{\frac{ABC}{MP^2}}$ (1) | Lower bound is same as (3) | Lower bound is higher than (6) | $\mathcal{D} = \frac{A}{P} + \frac{AB}{MP^2} + \frac{2C}{P}$ $\mathcal{V} = 2\sqrt{\frac{ABC}{MP^2}}$ (2) |
| *ACB* | Same as (1) | $\mathcal{D} = \frac{A}{P} + \frac{B}{P} + \frac{2AC}{MP^2}$ $\mathcal{V} = \frac{AC}{MP\sqrt{P}} + \frac{B}{\sqrt{P}}$ (3) | Lower bound is higher than (6) | Lower bound is same as (2) |
| *BAC* | $\mathcal{D} = \frac{B}{P} + 3\sqrt{\frac{ABC}{MP^3}}$ $\mathcal{V} = 2\sqrt{\frac{ABC}{MP^2}}$ (4) | Lower bound is higher than (3) | Lower bound is same as (6) | $\mathcal{D} = \frac{B}{P} + \frac{AB}{MP^2} + \frac{2C}{P}$ $\mathcal{V} = 2\sqrt{\frac{ABC}{MP^2}}$ (5) |
| *BCA* | Same as (4) | Lower bound is higher than (3) | $\mathcal{D} = \frac{A}{P} + \frac{B}{P} + \frac{2BC}{MP^2}$ $\mathcal{V} = \frac{A}{\sqrt{P}} + \frac{BC}{MP\sqrt{P}}$ (6) | Lower bound is same as (5) |
| *CAB* | $\mathcal{D} = \frac{2C}{P} + 2\sqrt{\frac{ABC}{MP^3}}$ $\mathcal{V} = 2\sqrt{\frac{ABC}{MP^2}}$ (7) | $\mathcal{D} = \frac{2C}{P} + \frac{AC}{MP^2} + \frac{B}{P}$ $\mathcal{V} = \frac{AC}{MP\sqrt{P}} + \frac{B}{\sqrt{P}}$ (8) | $\mathcal{D} = \frac{2C}{P} + \frac{A}{P} + \frac{BC}{MP^2}$ $\mathcal{V} = \frac{A}{\sqrt{P}} + \frac{BC}{MP\sqrt{P}}$ (9) | $\mathcal{D} = \frac{2C}{P} + 2\sqrt{\frac{ABC}{MP^3}}$ $\mathcal{V} = 2\sqrt{\frac{ABC}{MP^2}}$ (10) |
| *CBA* | Same as (7) | Same as (8) | Same as (9) | Same as (10) |

Table 5: Communication and Disk Access Volume for the **Inside Rotation** pattern

involves disk operations. Therefore, the total communication volume $\mathcal{V} = \sqrt{P} \times (\frac{A \times Jt_1}{P} + \frac{B \times It_1 \times It_2}{P}) = (\frac{A \times Jt_1}{\sqrt{P}} + \frac{B \times It_1 \times It_2}{\sqrt{P}})$. Due to in-memory transfer there will not be any disk access as part of the communication. The total disk access volume $\mathcal{D} = \frac{A}{P} + \frac{B}{P} \times It_1 + 2 \times \frac{C}{P} \times Kt_1$. For simplicity in the calculations below, $\mathcal{D}$ will not include the component $\frac{A}{P}$, which is independent of the number of tiles.

First we will try to optimize $\mathcal{D}$ and $\mathcal{V}$ independently. To minimize the communication volume $\mathcal{V}$, $It_1$, $It_2$ and $Jt_1$ should be made 1. But this is not possible due to the constraint that the tiles of array *A*, *B*, and *C* should fit into memory. Again we assume that each of these array tiles occupies one-third of memory. The constraints involving tiles are the same as those shown in the rotation case of **outside communication**.

Note that only Equation 4.3 involves all the variables whose values we want to reduce namely $It_1$, $It_2$, and $Jt_1$. We will try to minimize $\mathcal{V}$ under the constraint of Equation 4.3. The other two equations can be simultaneously satisfied by using a large value for the unconstrained variable $Kt_1$. Since we are trying to reduce the values of $It_1$, $It_2$, and $Jt_1$ while satisfying Equation 4.3, the Equation 4.3 can be written as $It_1 \times It_2 \times Jt_1 = \frac{C}{M \times P}$. With this modification, we can substitute the value of $Kt_1$ in the equation for $\mathcal{V}$ to get

$$B \times (It_1 \times It_2)^2 - \sqrt{P} \times \mathcal{V} \times (It_1 \times It_2) + \frac{A \times C}{M \times P} = 0 \quad (5.1)$$

The above quadratic equation will have a real solution when the condition, quadratic curve discriminant $P \times \mathcal{V}^2 - 4 \times B \times (\frac{A \times C}{M \times P}) \geq 0$ is true. In other words, for any real value of $It_1$ the minimum achievable value of $\mathcal{V}$ is $\frac{2}{P}\sqrt{\frac{A \times B \times C}{M}}$. This minimum value of $\mathcal{V}$ can be achieved with $It_1 = \frac{I}{\sqrt{M \times P}}$, $It_2 = 1$, $Jt_1 = \frac{J}{\sqrt{M \times P}}$, and $Kt_1 \geq \frac{K}{\sqrt{M \times P}}$ which also satisfies the Equations 4.1 and 4.2. Hence, the minimum total communication volume is

$$\frac{2}{P}\sqrt{\frac{A \times B \times C}{M}} \quad (5.2)$$

Now we will minimize the disk access volume independently. Note that $It_1$ and $Kt_1$ should be made 1 in this case. But this is not possible due to the constraint that the tiles of arrays *A*, *B*, and *C* should fit into the memory. We will try to minimize $\mathcal{D}$ under the constraint of Eqn. 4.1. The other two equations can be simultaneously satisfied by using a large value for the unconstrained variables $It_2$ and $Jt_1$. The Eqn. 4.1 in this case can be written as $It_1 \times Kt_1 = \frac{A}{M \times P}$. With this modification, we can substitute the value of $Kt_1$ in the equation for $\mathcal{D}$ to get

$$B \times It_1^2 - P \times \mathcal{D} \times It_1 + \frac{2 \times C \times A}{M \times P} = 0 \quad (5.3)$$

From this equation we can see that for any real value of $It_1$ the minimum achievable value of $\mathcal{D}$ is $\sqrt{\frac{8 \times A \times B \times C}{M \times P^3}}$. This minimum value of $\mathcal{D}$ can be achieved with $It_1 = I \times \sqrt{\frac{2}{M \times P}}$ and $Kt_1 = \frac{K}{\sqrt{2 \times M \times P}}$, $Jt_1 \geq J \times \sqrt{\frac{2}{M \times P}}$, and $It_2 \geq 1$. These values will also satisfy Equations 4.2 and 4.3. Hence, the minimum total disk access volume is

$$\frac{A}{P} + \sqrt{\frac{8 \times A \times B \times C}{M \times P^3}} \quad (5.4)$$

But it is obvious that the number of tiles does not match with that of the previous analysis to minimize communication volume. So we cannot optimize both the communication volume and disk access volume at the same time. We have computed the Overhead cost for both the cases and we choose the one which has the smaller Overhead cost. In this case we choose

| Disk Order | Cost Estimate Formulas | Tile Constraints | Minimal Disk Cost and conditions |
|---|---|---|---|
| ABC/<br>ACB | $\mathcal{V} = A$<br>$\mathcal{D} = \frac{A}{P} + \frac{B}{P} \times It1 + 2 \times \frac{C}{P} \times Kt1$ | $It1 \times Kt1 \geq \frac{A}{M}$<br>$Jt1 \times Kt1 \geq \frac{B}{M\times P}$<br>$It1 \times It2 \times Jt1 \geq \frac{C}{M\times P}$ | $<1>$ If $I \geq \sqrt{\frac{M}{2}}$, and $K \geq \sqrt{2M}$, then<br>$It1 = I\sqrt{\frac{2}{M}}, Kt1 = \frac{K}{\sqrt{2M}}, Jt1 \geq J\frac{\sqrt{2}}{\sqrt{MP}}, It2 = 1$<br>$\mathcal{D} = \frac{A}{P} + \frac{\sqrt{8ABC}}{\sqrt{MP}}$<br><br>$<2>$ If $I < \sqrt{\frac{M}{2}}$, and $K \geq \sqrt{2M}$, then<br>$It1 = 1, Kt1 = \frac{A}{M}, Jt1 \geq \frac{J}{I\times P}, It2 = 1$<br>$\mathcal{D} = \frac{A}{P} + \frac{B}{P} + \frac{2AC}{MP}$<br><br>$<3>$ If $I \geq \sqrt{\frac{M}{2}}$, and $K < \sqrt{2M}$, then<br>$It1 = \frac{A}{M}, Kt1 = 1, Jt1 \geq \frac{B}{M\times P}, It2 = 1$<br>$\mathcal{D} = \frac{A}{P} + \frac{AB}{MP} + \frac{2C}{P}$ |
| BAC/<br>BCA | $\mathcal{V} = A \times Jt1$<br>$\mathcal{D} = \frac{A}{P} \times Jt1 + \frac{B}{P} + 2 \times \frac{C}{P} \times Kt1$<br>$EffVol = A \times Jt1 \times \frac{1+PR}{P}$<br>$+\frac{B}{P} + 2 \times \frac{C}{P} \times Kt1$ | $It1 \times Kt1 \geq \frac{A}{M}$<br>$Jt1 \times Kt1 \geq \frac{B}{M\times P}$<br>$It1 \times Jt1 \times Jt2 \geq \frac{C}{M\times P}$ | $<1>$ If $J \geq \sqrt{\frac{MP(1+PR)}{2}}$, and $K \geq \sqrt{\frac{2MP}{1+PR}}$, then<br>$It1 \geq I\sqrt{\frac{2P}{M(1+PR)}}, Kt1 = K\sqrt{\frac{1+PR}{2MP}}, Jt1 = J\sqrt{\frac{2}{MP(1+PR)}}, Jt2 = 1$<br>$EffVol = \frac{B}{P} + \sqrt{\frac{8ABC(1+PR)}{MP^3}}$<br><br>$<2>$ If $J < \sqrt{\frac{MP(1+PR)}{2}}$, and $K \geq \sqrt{\frac{2MP}{1+PR}}$, then<br>$Jt1 = 1, Kt1 = \frac{B}{M\times P}, It1 \geq \frac{IP}{J}, Jt2 = 1$<br>$EffVol = \frac{B}{P} + \sqrt{\frac{8ABC(1+PR)}{MP^3}}$<br><br>$<3>$ If $J \geq \sqrt{\frac{MP(1+PR)}{2}}$, and $K < \sqrt{\frac{2MP}{1+PR}}$, then<br>$Jt1 = \frac{B}{M\times P}, Kt1 = 1, It1 \geq \frac{A}{M}, Jt2 = 1$<br>$EffVol = \frac{B}{P} + \frac{A(1+PR)}{P} + \frac{2BC}{MP^3}$ |
| CAB/<br>CBA | $\mathcal{V} = A \times Jt1$<br>$\mathcal{D} = \frac{A}{P} \times Jt1 + \frac{B}{P} \times It1 + 2 \times \frac{C}{P}$<br>$EffVol = A \times Jt1 \times \frac{1+PR}{P}$<br>$+\frac{B}{P} \times It1 + 2 \times \frac{C}{P}$ | $It1 \times Kt1 \geq \frac{A}{M}$<br>$Jt1 \times Jt2 \times Kt1 \geq \frac{B}{M\times P}$<br>$It1 \times Jt1 \geq \frac{C}{M\times P}$ | $<1>$ If $I \geq \sqrt{\frac{MP}{1+PR}}$, and $J \geq \sqrt{MP(1+PR)}$, then<br>$It1 = I\sqrt{\frac{1+PR}{M}}, Kt1 \geq K\sqrt{\frac{P}{M(1+PR)}}, Jt1 = J\sqrt{\frac{J}{MP(1+RP)}}, Jt2 = 1$<br>$EffVol = \frac{2C}{P} + \sqrt{\frac{4ABC(1+RP)}{MP^3}}$<br><br>$<2>$ If $I < \sqrt{\frac{MP}{1+PR}}$, and $J \geq \sqrt{MP(1+PR)}$, then<br>$It1 = 1, Jt1 = \frac{C}{M\times P}, Kt1 \geq \frac{A}{M}, Jt2 = 1$<br>$EffVol = \frac{2C}{P} + \frac{B}{P} + \frac{AC(1+PR)}{MP^2}$<br><br>$<3>$ If $I \geq \sqrt{\frac{MP}{1+PR}}$, and $J < \sqrt{MP(1+PR)}$, then<br>$It1 = \frac{C}{M\times P}, Jt1 = 1, Kt1 \geq \frac{KP}{J}, Jt2 = 1$<br>$EffVol = \frac{2C}{P} + \frac{BC}{MP^2} + A(1+PR)$ |

Table 6: Communication and Disk Access Volume and Conditions for the **Inside/Replication** pattern

| Disk Order | Cost Estimate Formulas | Tile Constraints | Minimal Disk Cost and conditions |
|---|---|---|---|
| ABC/<br>ACB | $\mathcal{V} = C \times LogP \times Kt1$<br>$\mathcal{D} = \frac{A}{P} + \frac{B}{P} \times It1 + 2C \times Kt1$<br>$EffVol = \frac{A}{P} + \frac{B}{P} \times It1$<br>$+C \times Kt1 \times (2 + R \times logP)$ | $It1 \times Kt1 \geq \frac{A}{M \times P}$<br>$Jt1 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times It2 \times Jt1 \geq \frac{C}{M}$ | $< 1 >$ If $I \geq \sqrt{\frac{M}{2+RlogP}}$, and $K \geq P\sqrt{M(2+RlogP)}$, then<br>$It1 = I\sqrt{\frac{(2+RlogP)}{M}}$, $Kt1 = \frac{K}{\sqrt{(2+RlogP)MP}}$, $Jt1 \geq J\sqrt{\frac{2+RlogP}{M}}$, $It2 = 1$<br>$EffVol = \frac{A}{P} + 2\sqrt{\frac{ABC(2+RlogP)}{MP^2}}$<br><br>$< 2 >$ If $I < \sqrt{\frac{M}{2+RlogP}}$, and $K \geq P\sqrt{M(2+RlogP)}$, then<br>$It1 = 1$, $Kt1 = \frac{A}{M \times P}$, $Jt1 \geq \frac{J}{I}$, $It2 = 1$<br>$EffVol = \frac{A}{P} + \frac{B}{P} + \frac{AC}{MP} \times (2 + RlogP)$<br><br>$< 3 >$ If $I \geq \sqrt{\frac{M}{2+RlogP}}$, and $K < P\sqrt{M(2+RlogP)}$, then<br>$It1 = \frac{A}{MP}$, $Kt1 = 1$, $Jt1 \geq \frac{B}{M \times P}$, $It2 = 1$<br>$EffVol = \frac{A}{P} + \frac{AB}{MP} + C(2 + RlogP)$ |
| BAC/<br>BCA | $\mathcal{V} = C \times LogP \times Kt1$<br>$\mathcal{D} = \frac{A}{P} \times Jt1 + \frac{B}{P} + 2C \times Kt1$<br>$EffVol = \frac{A}{P} \times Jt1 + \frac{B}{P}$<br>$+C \times (2 + RlogP) \times Kt1$ | $It1 \times Kt1 \geq \frac{A}{M \times P}$<br>$Jt1 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times Jt1 \times Jt2 \geq \frac{C}{M}$ | $< 1 >$ If $J \geq \sqrt{\frac{M}{2+RlogP}}$, and $K \geq \sqrt{MP^2(2+RlogP)}$, then<br>$It1 \geq I\sqrt{\frac{P(2+RlogP)}{M}}$, $Kt1 = \frac{K}{\sqrt{MP^2(2+RlogP)}}$, $Jt1 = J\sqrt{\frac{(2+RlogP)}{M}}$, $Jt2 = 1$<br>$EffVol = \frac{B}{P} + 2\sqrt{\frac{ABC(2+RlogP)}{MP^2}}$<br><br>$< 2 >$ If $J < \sqrt{\frac{M}{2+RlogP}}$, and $K \geq \sqrt{MP^2(2+RlogP)}$, then<br>$Jt1 = 1$, $Kt1 = \frac{B}{M \times P}$, $It1 \geq \frac{I}{J}$, $Jt2 = 1$<br>$EffVol = \frac{A}{P} + \frac{B}{P} + \frac{BC}{MP} \times (2 + RlogP)$<br><br>$< 3 >$ If $J \geq \sqrt{\frac{M}{2+RlogP}}$, and $K < \sqrt{MP^2(2+RlogP)}$, then<br>$Jt1 = \frac{B}{M \times P}$, $Kt1 = 1$, $It1 \geq \frac{A}{M \times P}$, $Jt2 = 1$<br>$EffVol = \frac{B}{P} + \frac{AB}{MP} + C \times (2 + RlogP)$ |
| CAB/<br>CBA | $\mathcal{V} = C \times logP$<br>$\mathcal{D} = \frac{A}{P} \times Jt1 + \frac{B}{P} \times It1 + 2C$<br>$EffVol = A \times Jt1 \times + \frac{B}{P} \times It1$<br>$+C \times (2 + RlogP)$ | $It1 \times Kt1 \geq \frac{A}{M \times P}$<br>$Jt1 \times Jt2 \times Kt1 \geq \frac{B}{M \times P}$<br>$It1 \times Jt1 \geq \frac{C}{M}$ | $< 1 >$ If $I \geq \sqrt{M}$, and $J \geq \sqrt{M}$, then<br>$It1 = \frac{I}{\sqrt{M}}$, $Kt1 \geq \frac{K}{\sqrt{MP^2}}$, $Jt1 = \frac{J}{\sqrt{M}}$, $Jt2 = 1$<br>$EffVol = C \times (2 + RlogP) + \sqrt{\frac{4ABC}{MP^2}}$<br><br>$< 2 >$ If $I < \sqrt{M}$, and $J \geq \sqrt{M}$, then<br>$It1 = 1$, $Jt1 = \frac{C}{M}$, $Kt1 \geq \frac{A}{PM}$, $Jt2 = 1$<br>$EffVol = C \times (2 + RlogP) + \frac{B}{P} + \frac{AC}{MP}$<br><br>$< 3 >$ If $Igeq\sqrt{M}$, and $J < \sqrt{M}$, then<br>$It1 = \frac{C}{M}$, $Jt1 = 1$, $Kt1 \geq \frac{B}{MP}$, $Jt2 = 1$<br>$EffVol = C \times (2 + RlogP) + \frac{A}{P} + \frac{BC}{MP}$ |

Table 7: Communication and Disk Access Volume and Conditions for the **Inside/Accumulation** pattern

the number of tiles that optimizes the communication volume as this gives the least Overhead cost. The values of communication and disk access volume are as follows with these tile sizes:

$$\mathcal{V} = \frac{2}{P}\sqrt{\frac{A \times B \times C}{M}} \quad (5.5)$$

$$\mathcal{D} = \frac{A}{P} + 3\sqrt{\frac{A \times B \times C}{M \times P^3}} \quad (5.6)$$

There are three special cases if values of $It_1$, $Jt_1$, or $Kt_1$ are less than 1.

- *Case 1:* $I < \sqrt{M \times P}, J \geq \sqrt{M \times P}, K \geq \sqrt{M \times P}$, In this case, the expected least overhead is $\mathcal{V} = \frac{A \times C}{\sqrt{M \times P^3}} + \frac{B}{\sqrt{P}}$ and $\mathcal{D} = \frac{A}{P} + \frac{B}{P} + \frac{2 \times C \times A}{M \times P^2}$ with $It_1 = 1$, $It_2 = 1$, $Jt_1 = \frac{C}{M \times P}$, $Kt_1 = \frac{A}{M \times P}$. But with these values, Eqn. 4.2 is not satisfied. So, the least overhead above can not be really achieved. This is not a problem, though, as the expected lower bound in this case is same as the achievable lower bound of case ACB as shown in Table 5.
- *Case 2:* $I \geq \sqrt{M \times P}, J < \sqrt{M \times P}, K \geq \sqrt{M \times P}$, the expected least overhead is $\mathcal{V} = \frac{A}{\sqrt{P}} + \frac{B \times C}{\sqrt{M \times P^3}}$ and $\mathcal{D} = \frac{A}{P} + 3 \times \sqrt{\frac{ABC}{M \times P^3}}$ with $It_1 = 1, It_2 = 1, Jt_1 = \frac{C}{M \times P}, Kt_1 = \frac{A}{M \times P}$. But with these values, Eqn. 4.3 is not satisfied. So, the least overhead above can not be really achieved. We don't need to mind it, from Table 5, we can see that the achievable lower bound of case BCA is $\mathcal{V}' = \frac{A}{\sqrt{P}} + \frac{B \times C}{\sqrt{M \times P^3}}$ and $\mathcal{D}' = \frac{A}{P} + \frac{A}{P} + 2 \times \frac{BC}{M \times P}$, which is lower than the expected lower bound in the current case.
- *Case 3:* $I \geq \sqrt{M \times P}, J \geq \sqrt{M \times P}, K < \sqrt{M \times P}$,

  In this case, the least overhead that can be achieved is $\mathcal{V} = 2\sqrt{\frac{A \times B \times C}{M \times P^2}}$ and $\mathcal{D} = \frac{A}{P} + \frac{B}{P} \times \frac{A}{M \times P} + \frac{2 \times C}{P}$ with $It_1 = \frac{A}{M \times P}$, $It_2 = \frac{\sqrt{M \times P}}{K}$, $Jt_1 = \frac{J}{\sqrt{M \times P}}$, $Kt_1 = 1$. With these values, all the constraints are also satisfied.

We did the analysis for the other five disk placement orders as above. The results of the analysis are shown in Table 5.

## 5.2 Replication

For this case, let us consider the tensor contraction code with disk I/O placement order *ABC*, an inside communication pattern, and the replication type of parallelism as shown in Figure 1(b).

In this case, because the replication occurs in memory, and replicated data will be skipped after computation, so array $A$ is not replicated on disk. Arrays $A$, $B$ and $C$ are partitioned equally among the $P$ processors. We have $A.localsize = A/P$, $B.localsize = B/P$, $C.localsize = C/P$. The communication corresponds to an in-core broadcast of array $A$. Therefore, the total communication volume $\mathcal{V} = A$, and it is independent of the tile sizes. The total disk access volume $\mathcal{D} = \frac{A}{P} + \frac{B}{P} \times It_1 + 2 \times \frac{C}{P} \times Kt_1$.

The constraints involving tiles are the same as those shown in the replication part of outside communication. We do the analysis similar to the ones for the earlier cases. The minimum achievable value of $\mathcal{D}$ can be computed as $\frac{A}{P} + \frac{1}{P}\sqrt{\frac{8 \times A \times B \times C}{M}}$. This minimum value of $\mathcal{D}$ can be achieved with $It_1 = I \times \sqrt{\frac{2}{M}}$ and $Kt_1 = \frac{K}{\sqrt{2 \times M}}$, $Jt_1 \geq \frac{J}{P} \times \sqrt{\frac{2}{M}}$ and $It_2 \geq 1$. These values satisfy all the constraints. The analysis for the special cases can be done in the earlier sections.

The result of the analysis for the other five disk placement orders are shown in Table 6. Note that the values shown in this table are the effective communication and disk access volume $EffVol = \mathcal{D} + R \times \mathcal{V}$, where $R = \frac{B_d}{B_c}$, where $B_d$ is the disk bandwdith and $B_c$ is the communication (network) bandwidth.

## 5.3 Accumulation

In this section, we deal with the accumulation type of parallelism. Consider the tensor contraction code with the accumulation type of parallelism as shown in Figure 1(c). In this case, arrays $A$ and $B$ are partitioned equally among the $P$ processors whereas $C$ is replicated on all processors. So we have $A.localsize = A/P$, $B.localsize = B/P$, $C.localsize = C$. The communication involves in-core All-Reduce operation of array $C$. Therefore, the total communication volume $\mathcal{V} = C \times Kt_1 \times \log P$. The total disk access volume $\mathcal{D} = \frac{A}{P} + \frac{B}{P} \times It_1 + 2 \times C \times Kt_1$. In this case, we can optimize the total overhead cost, which is $\frac{EffVol}{B_d}$, where *EffVol* is the effective communication and disk access volume given by (note that $R$ is defined at the end of Section 5.2)

$$EffVol = \frac{A}{P} + \frac{B}{P} \times It_1 + C \times Kt_1 \times (2 + R \times \log P). \quad (5.7)$$

Our goal is to minimize *EffVol* under the constraints involving tile sizes that are shown in the accumulation section of the previous section. We proceed as before and compute the minimum achievable value of *EffVol*, which is found to be $\frac{A}{P} + 2 \times \sqrt{\frac{ABC(2+R \times \log P)}{M \times P^2}}$. This minimum value is achieved with $It_1 = I \times \sqrt{\frac{(2+R \times \log P)}{M}}$, $Kt_1 = \frac{K}{P \times \sqrt{(2+R \times \log P) \times M}}$, $Jt_1 \geq J \times \sqrt{\frac{(2+R \times \log P)}{M}}$, and $It_2 \geq 1$.

The special cases are handled as before. The analysis for the other five disk placement orders are also done as above. The results are shown in Table 7. Again, note that the values in the table give the minimum value of *EffVol*.

# 6 Experiments

Our performance models for the various approaches to parallel out-of-core tensor contractions were evaluated on an Itanium-2 cluster at the Ohio Supercomputer Center. The configuration of the cluster is shown in Table 8. All the programs were compiled with the Intel Itanium Fortran Compiler for Linux. We considered three example computations.

**(1) Square Matrix Multiplication:**

$$C(I,J) += A(I,K) \times B(J,K) \quad (6.1)$$

In order to limit the execution time we ran "scaled down" experiments by setting the available physical memory limit to 64Mbytes. All the array dimensions were set to 4000. The parallel programs were run on 4 processors. We implemented parallel programs for the six methods discussed earlier. Table 9 compares the predicted costs for I/O and communication

| Node | OS | Compiler | Memory | Network Bandwidth | Disk Bandwidth |
|---|---|---|---|---|---|
| Dual 900MHz | Linux | efc | 1GB | 200MB/s | 8MB/s |

Table 8: Configuration of the Itanium 2 cluster at OSC

with the measured costs for the different approaches. It can be seen that there is a good match between predicted and actual times, and that the difference in performance of the various methods is quite significant.

**(2) 4-index transform:** This expression (also referred to as the AO-to-MO transform) is commonly used to transform two-electron integrals from an atomic orbital (AO) basis to a molecular orbital (MO) basis.

$$T1[a,b,c,d] += A[a,b,c,p] \times B[p,d] \tag{6.2}$$

The size of all dimensions was set to 800. The parallel program was run on 4 processors. Between the different algorithms, we can find the best solution to be outside replication. The predicted overheads for the different parallel algorithms are shown in Table 10.

**(3) CCSD:** We used a sub-expression from the CCSD (Coupled Cluster Singles and Doubles) model [1, 19, 20] for determine electronic structures.

$$T1[i,j] += A[i,a,b,c] \times B[a,b,c,j] \tag{6.3}$$

The size of all dimensions was set to 800. The parallel program was run on 4 processors. The best solution on the current machine can be seen to be outside accumulation. The predicted values of different parallel algorithms are shown in Table 10.

The effective choice of parallel algorithms results in a noticeable improvement in the communication cost for most cases. The ratio of disk bandwidth and interprocessor network bandwidth determines which factor dominates the total execution time. In previous experiments, because the network is almost twenty times faster than the disk, the disk cost dominated. In such a situation, the inside rotation algorithm is the best. However, using our model, we are able to predict the best choice for a given machine and problem characteristic. Table 11 shows such an example for a given matrix multiplication and disk bandwidth, where $I = 160000$, $J = 160000$, $K = 160000$, and $B_d = 10MB/s$. If we use a 100M Ethernet as the interconnection network and run the program on 4 processors, then the best parallel algorithm is outside replication. If we use Myrinet and run the program on 4 processors, the best solution becomes inside rotation.

## 7 Related Work

The issues arising in optimizing locality in the context of tensor contractions has been previously addressed by us, focusing primarily on minimizing memory-to-cache data movement [8, 9]. This approach was extended to the disk-memory hierarchy in [15], where a greedy approach to disk read/write placement was taken. For each set of tile sizes, the algorithm places read/write statements immediately inside those loops at which the memory limit is exceeded. In [16], a set of candidate fusion structures with disk I/O placements was taken as input and the tile size search space was explored. The search space was divided into feasible and infeasible solution spaces and their boundary was shown to contain the optimal solution. An algorithm was developed to locate the boundary efficiently and a steepest ascent hill-climbing used to determine an efficient solution for the tile sizes.

There has been some work in the area of software techniques for optimizing disk I/O. These include parallel file systems, compile time [4–6, 11–13, 17, 18] and runtime libraries and optimizations [7, 22]. Bordawekar et al. [4, 5] discuss several compiler methods for optimizing out-of-core programs in High Performance Fortran. Bordawekar et al. [6] develop a scheduling strategy to eliminate additional I/O arising from communication among processors; this paper is among the few that address the impact of scheduling on disk I/O overhead in a parallel context. Solutions for choreographing disk I/O with computation are presented by Paleczny et al. [18]. They organize computations into groups that operate efficiently on data accessed in chunks. Compiler-directed prefetching is discussed by Mowry et al. [17]. ViC* (Virtual C*) [10] is a preprocessor that transforms out-of-core C* programs into in-core programs with appropriate calls to the ViC* I/O library. Kandemir et al. [11–13] develop file layout and loop transformations for reducing I/O. None of these techniques address performance modeling and optimization of of parallel out-of-core computations addressing both disk I/O costs and interprocessor communication overheads.

There has been some work in the design of out-of-core linear algebra libraries [23, 24]. But, we are not aware of any work that addresses the detailed modeling of disk I/O and inter-processor communication costs, in addition to an evaluation and optimization of the overall performance of parallel out-of-core computations.

## 8 Conclusion and Future Work

This paper addressed the problem of developing performance models for a core computation – tensor contractions – in the context of a domain-specific compiler targeting a class of computations in quantum chemistry. The cost of disk I/O and interprocessor communication for different data partitions and tile sizes was modeled for various computational alternatives. The models were experimentally evaluated and the predictions were shown to match measured results. It was also seen that the optimal choice of parallel algorithm was dependent on the characteristics of both the tensor structure as well as machine parameters. Further work is in progress for using this framework to optimize tensor contraction expressions with a sequence of tensor contractions.

## Acknowledgments

We would like to thank the Ohio Supercomputer Center (OSC) for the use of their computing facilities.

| Parallel Method | Predicted Overhead (sec.) | Measured Overhead (sec.). |
|---|---|---|
| Inside/Rotation | 25.28 | 28.8827 |
| Outside/Rotation | 75.42 | 70.1237 |
| Inside/Replication | 24.718 | 23.5525 |
| Outside/Replication | 56.9 | 63.1590 |
| Inside/Accumulation | 53.792 | 54.7575 |
| Outside/Accumulation | 73.792 | 78.6768 |

Table 9: Predicted and Empirical results

| | 4index | | | ccsd | | |
|---|---|---|---|---|---|---|
| | Disk I/O Volume(MB) | Comm. Volume(MB) | Total Time(sec.) | Disk I/O Volume(MB) | Comm. Volume(MB) | Total Time(sec.) |
| Outside/Rotation | 1024000.32 | 204800.32 | 829440 | 1024000 | 409600 | 839680 |
| Outside/Replica. | 307200.64 | 0.64 | 245760 | 512000 | 409600 | 438563 |
| Outside/Accumula. | 921600.16 | 819200 | 778240 | 204800.64 | 1.28 | 163840 |
| Inside/Rotation | 307200.16 | 205824.32 | 256051 | 204800.16 | 409600 | 184320 |
| Inside/Replica. | - | - | - | 921600.32 | 409600 | 766243 |
| Outside/Accumula. | 512000 | 1146880 | 466944 | - | - | |

Table 10: Predicted performance results on 4 processors for ccsd and 4index-transform

| | I=J=K=160000 , 4 Processors | | I = J = K =640000, 16 Processors | |
|---|---|---|---|---|
| | $B_c = 10MB/s$ | $B_c = 200MB/s$ | $B_c = 10MB/s$ | $B_c = 200MB/s$ |
| Outside/Rotation | 281920 | 262464 | **3855360** | 3699712 |
| Outside/Replication | **259683** | 232168 | 4224000 | 3601408 |
| Outside/Accumulation | 318784 | 264307 | 6018048 | 4274790 |
| Inside/Rotation | 310240 | **120240** | 4040960 | **1000960** |
| Inside/Replication | 268248 | 123502 | 4636610 | 1330921 |
| Inside/Accumulation | 298304 | 243827 | 5690368 | 3947110 |

Table 11: When $B_d$=10MB/s, predicted best disk/communication overheads (in sec.)

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