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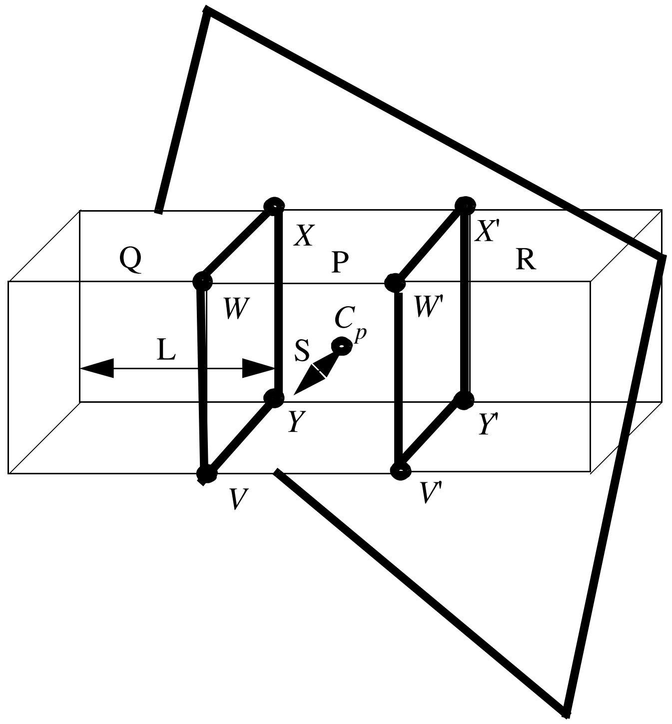

Figure 12. Proof of the 3D case for 6-separability and 6-minimality.

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But $t_{26} = (L/2)\sqrt{3}\cos\alpha$ so $S \le t_{26}$.In this case voxel P would have been included in the representation of the plane. Contradiction. So our initial assumption that there exists such a voxel P and that the plane has a 26-hole is false. So the plane is 26-separating.

Minimality:

Let us prove that this plane is minimal. Let us assume that their is a voxel in the representation of the plane that can be removed without introducing a 26-hole. For this voxel condition $S \le t_{26}$ must hold so the distance from the center of the voxel to the plane is $S \le (L/2)\sqrt{3}\cos\alpha$. That means that there are at least two vertices of this voxel such that they are on opposite sides of the plane. But in this case removing this voxel from the representation of the plane will introduce a 26-hole. So our representation of the plane is 26-minimal.

Let us now use the value of t_6 in place of t in equation (3).

Teorem 4.

The set of voxels $\{ (x, y, z) \mid -t_6 < A_p x + B_p y + C_p z + D_p \le t_6 \}$ gives 6-separating, 6-minimal plane.

Proof:

Separability:

Let us use the proof by contradiction.For our plane to have a 6-hole there must be a voxel that is not included in the representation of the plane and that has at least one 6 neighbor on one side of the plane and another 6-neighbor on the other side of the plane.

Let P be such a voxel and Q and R be it's neighbors, as shown in Figure 8. In this case edges VW and V'W' are on different sides of the plane. The distance between the center of the voxel P, denoted by C_p, and the face V'W'X'Y' is $(L/2)$ and the distance between the center of the voxel and the plane is $S \le (L/2)\sqrt{2}\cos\beta$. Where β is an angle between the normal to the plane and VV'W'W. But $t_6 = (L/2)\sqrt{2}\cos\beta$ so $S \le t_6$.In this case voxel P would have been included in the representation of the plane. Contradiction. So our initial assumption that there exists such a voxel P and the plane has an 6-hole is false. So the plane is 6-separating.

Minimality:

Let us prove that this plane is minimal. Let us assume that their is a voxel in the representation of the plane that can be removed without introducing an 6-hole. For this voxel condition $S \le t_6$ must hold so the distance from the center of the voxel to the line is $S \le (L/2)\sqrt{2}\cos\beta$. That means that there are at least two faces of this voxel such that they are on the opposite sides of the plane. But in this case removing this voxel from the representation of the plane will introduce a 6-hole. So our representation of the plane is 6-minimal.

Teorem 3.

The set of voxels $\{ (x, y, z) \,|\, -t_{26} < A_p x + B_p y + C_p z + D_p \le t_{26} \}$ gives 26-separating, 26-minimal plane.

Proof:

Separability:

Let us use the proof by contradiction.For our plane to have a 26 hole there must be a voxel that is not included in the representation of the plane and that has at least one 26 neighbor on one side of the plane and another 26 neighbor on the other side of the plane.

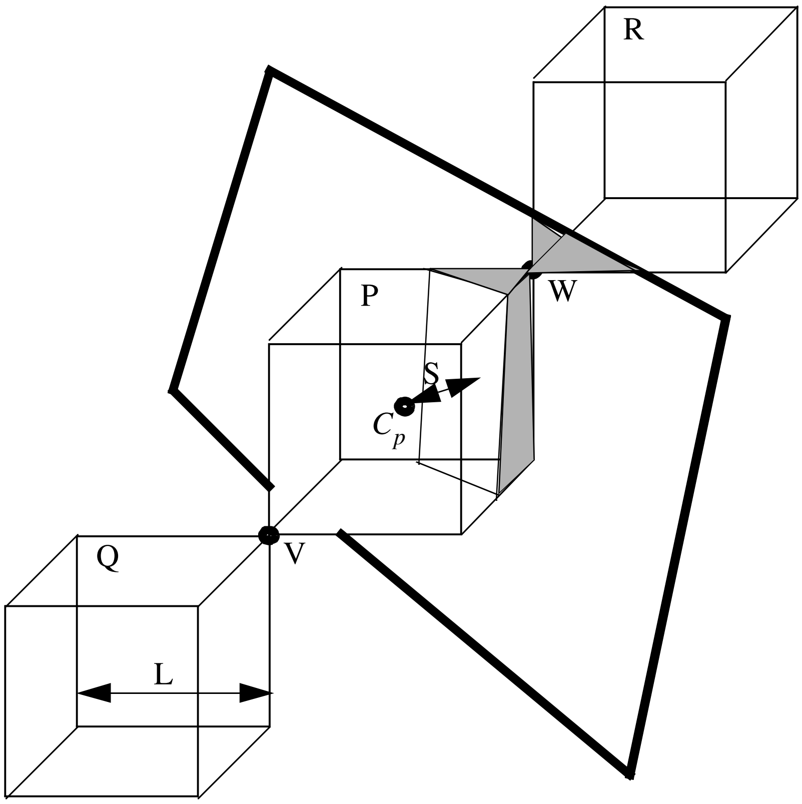


Figure 11. Proof of the 3D case for 26-separability and 26-minimality.

Let P be such a voxel and Q and R be it's neighbors, as shown in Figure 8. In this case points V and W are on the different sides of the plane. The distance between the center of the voxel P, denoted by C_p, and point V is $(L/2)\sqrt{3}$ and the distance between the center of the voxel and the plane is $S \le (L/2)\sqrt{3}\cos\alpha$. Where α is an angle between the normal to the plane and VW.

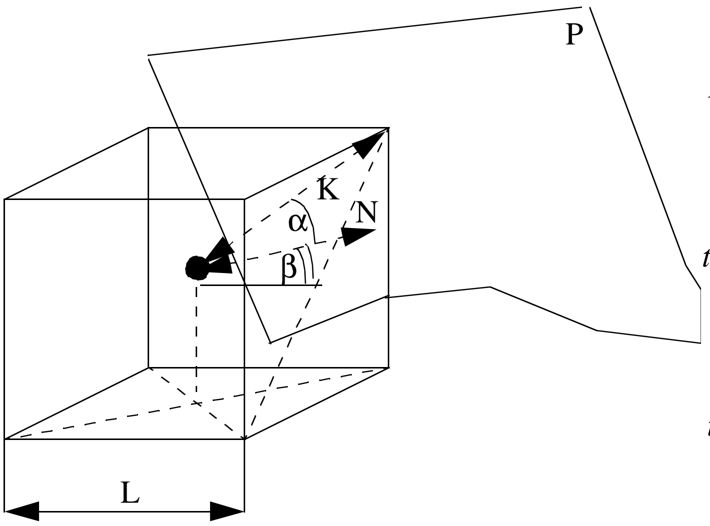
separating.

Minimality:

Let us prove that this line is minimal. Let us assume that their is a pixel in the representation of the line that can be removed without introducing an 4-hole. For this pixel condition $S \leq t_4$ must hold so the distance from the center of the pixel to the line is $S \leq (L/2)\cos\beta$. That means that there are at least two edges of this pixel such that they are on opposite sides of the line. But in this case removing the pixel P from the representation of the line will introduce a 4-hole. So our representation of the line is 4-minimal.

4.2 Voxelizing a 3D Plane

In the 3D case we will have to use equation (3) instead of equation 4 and lines G and H will be replaced by planes G and H. The decision whether to include a given voxel into the digital representation of the polygon is handled the same way as in 2D. If the center of the voxel is between planes G and H then this voxel is included into the digital representation of the polygon. In 3D space we will be concerned with building either 6- or 26-separating representations of our polygons.



$$K_3 = (L/2)\sqrt{3}$$

$$t_{26} = K_3\cos\alpha = (L/2)\sqrt{3}\cos\alpha$$

$$t_6 = (L/2)\sqrt{2}\cos\beta$$

Figure 10. The 3D case: L is the voxel's width, and K is the diagonal. N is the normal to the line from the voxel's center. α is the angle between N and K.

Let us use the same reasoning as with 2D case to come up with distances between planes G and H which will give the representation of the polygon with desired separability.

Let us use the value of t_{26} in place of t in equation (3), as shown in Figure 10.

8-hole. So our representation of the line is 8-minimal.

Let us now use the value of t_4 in place of t in equation $-t < A_pX + B_pY + D_p \le t$.

Teorem 2.

The set of pixels $\{ (x, y) \mid -t_4 < A_px + B_py + D_p \le t_4\}$ gives 4-separating, 4-minimal line.

Proof:

Separability:

Let us use the proof by contradiction.For our line to have a 4-hole there must be a pixel that is not included in the representation of the line and that has at least one 4 neighbor on one side of the line and another 4 neighbor on the other side of the line.

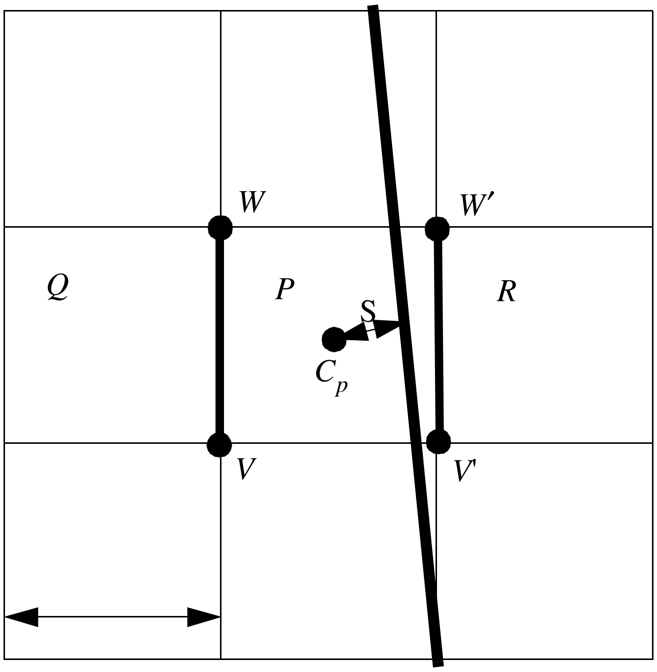


Figure 9. Proof of the 2D case for 4-separability and 4-minimality.

Let P be such a voxel and Q and R be its neighbors, as shown in Figure 9. Without loss of generality let us consider the case in the Fig. 4. In this case, edges VW and V'W' are on the different sides of the line. The distance between the center of the pixel P, denoted by C_p , and the edge V'W' is $(L/2)$ and the distance between the center of the pixel and the line is $S \le (L/2)\cos\beta$, where β is the angle between the normal to the line and VV'. But $t_4 = (L/2)\cos\beta$ so $S \le t_4$.In this case pixel P would have been included in the representation of the line. Contradiction. So our initial assumption that there exists such pixel P and the line has an 4-hole is false. So the line is 4-

Proof:

Separability:

Let us use the proof by contradiction. For our line to have an 8-hole there must be a pixel that is not included in the representation of the line and that has at least one 8-neighbor on one side of the line and another 8-neighbor on the other side of the line.

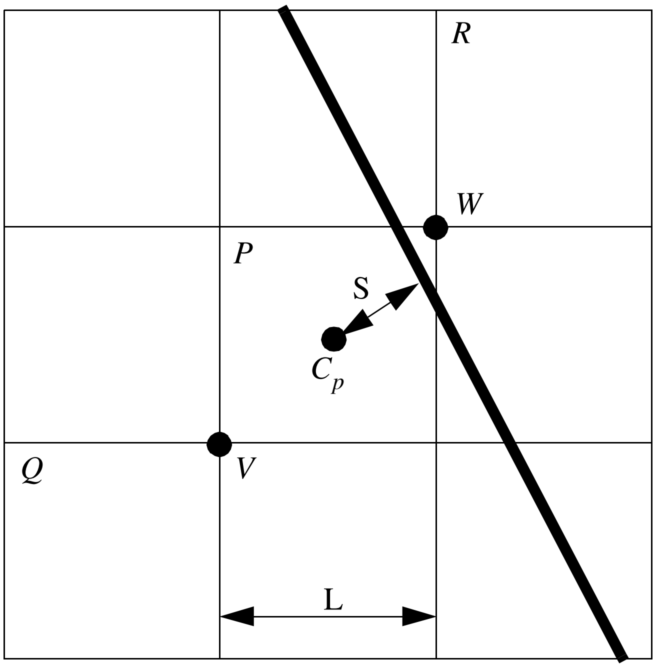


Figure 8. Proof of the 2D case for 8-separability and 8-minimality.

Let *P* be such a pixel and *Q* and *R* be it's neighbors, as shown in Figure 8. Without loss of generality let us consider the case in the Fig. 3. In this case points *V* and *W* are on the different sides of the line and the center of the pixel is on the same side of the line as point V. The distance between the center of the pixel *P*, denoted by C_p, and point *W* is $(L/2)\sqrt{2}$ and the distance between the center of the pixel and the line is $S \le (L/2)\sqrt{2}\cos\alpha$. Where α is an angle between the normal to the line and the diagonal VW. But $t_8 = (L/2)\sqrt{2}\cos\alpha$ so $S \le t_8$. In this case pixel *P* would have been included in the representation of the line. Contradiction. So our initial assumption that there exists such pixel *P* and that the line has an 8-hole is false. So the line is 8-separating.

Minimality:

Let us prove that this line is minimal. Let us assume that there is a pixel in the representation of the line that can be removed without introducing an 8-hole. For this pixel the condition $S \le t_8$ must hold so the distance from the center of the pixel to the line is $S \le (L/2)\sqrt{2}\cos\alpha$. That means that there are at least two vertices of this pixel such that they are on the opposite sides of the line. But in this case removing this pixel from the representation of the line will introduce an

Equations for the planes G and H will have the following form:

$$A_pX + B_pY + C_pZ + D_p \pm t = 0 \tag{2}$$

A voxel centered at the grid point $\{x, y, z\}$ will be incorporated in the representation of the plane P if the point $\{x, y, z\}$ lies between the planes G and H.

Point $\{x, y, z\}$ lies between planes G and H if the following condition holds:

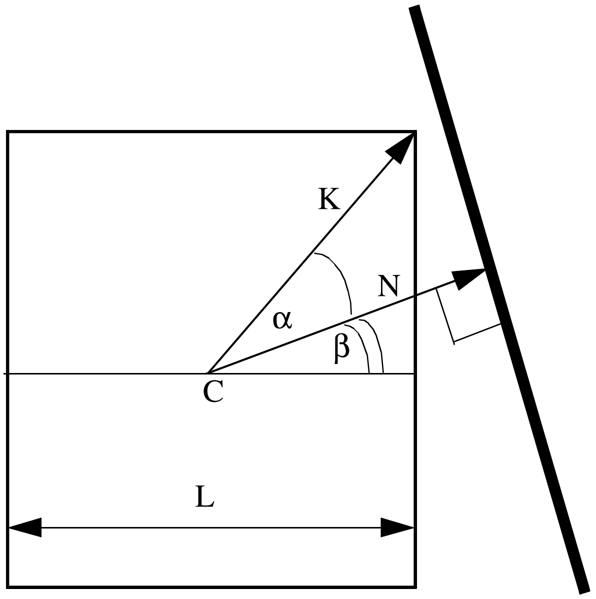
$$-t < A_pX + B_pY + C_pZ + D_p \le t \tag{3}$$

4.1 Pixelizing a 2D Line

In the 2D case we will deal with straight lines instead of plains and the conditions in the problem will require 4- and 8- separability and minimality. In that case we will have lines G and H instead of planes G and H and equation (3) will take the form:

$$-t < A_pX + B_pY + D_p \le t \tag{4}$$

Let us derive the distance between lines G and H corresponding to 8-separating line and 4-separating line. In the equations below L is the size of the grid square (pixel), as shown in Figure 7. C is the pixel's center and N is the normal to the line drawn from the point C.



$$K_2 = (L/2)\sqrt{2}$$

$$t_8 = K_2 \cos\alpha = (L/2)\sqrt{2}\cos\alpha$$

$$t_4 = (L/2)\cos\beta$$

Figure 7. The 2D case: L is the pixel's width, C denotes its center, and K is the diagonal. N is the normal to the line from the pixel's center. α is the angle between N and K.

Let us use the value of t_8 in place of t in equation $-t < A_pX + B_pY + D_p \le t$.

Teorem 1.

The set of pixels $\{ (x, y) \mid -t_8 < A_px + B_py + D_p \le t_8 \}$ is an 8-separating, 8-minimal line.

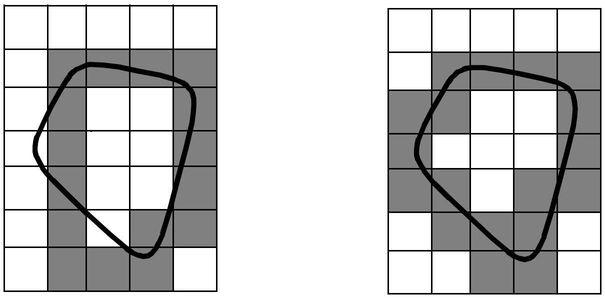


Figure 5. Two 4-separating with different accuracies; the pixelization in the right seems to better approximate the continuous curve then the one on the left.

Finally, in addition to separability and minimality, $\tilde{S}$ is required to approximate S as closely as possible or. For example, the two voxelization of the curve in Figure 5 are both 4-separating in 2D space, but the right one seems to better approximate the curve. The mechanisms that assess the accuracy of $\tilde{S}$ depends on the specific algorithm and involves numerical considerations that are discussed later.

4. Voxelization of Lines and Planes

Given a polygon in 3D space, an integer number N where $\forall \in \{6, 26\}$ and a uniform grid in 3D space construct a digital representation of the polygon that is N-separating and N-minimal.
We note that separability is a topological manifestation of thickness. We will use actual thickness to control separability.
Given a plane P let's construct two parallel planes such that our plane lies between these planes and is parallel to each of them, as shown in Figure 6.

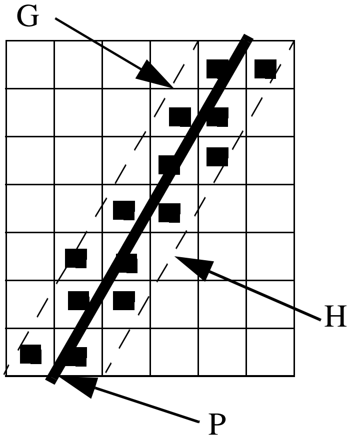


Figure 6. The plane P (solid line) and the two planes parallel to P at distance t, G and H (dashed lines). All voxels between G and H are included in the discrete representation of P.

Given an equation for plane P:

$$A_pX + B_pY + C_pZ + D_p = 0 \tag{1}$$

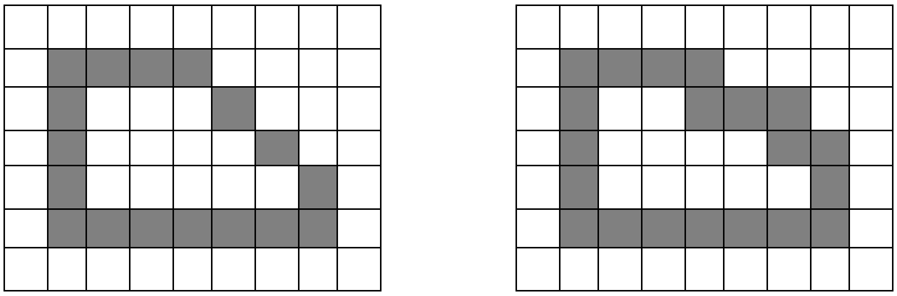


Figure 3. A 4-separating curve (left) and an 8-separating curve (right).

Let $S \in R^3$ be a continuous surface such that $R^3 - S$ has exactly two connected components I and O. Let $\tilde{S}$ be a voxelization of S, if $\Sigma - \bar{S}$ has exactly two N-components I and O, then $\tilde{S}$ is said to be *N-separating with respect to S*, or, in short, *N-separating*. This property is topological and does not reflect on how close $\tilde{S}$ is to the continuous surface S. In practice, in order to avoid dealing with the special cases of surface boundaries, we regard N-separability as a local surface property, for example, on a $3 \times 3 \times 3$ neighborhood.

A voxel belonging to a N-separating surface is called *N-simple voxel* if its deletion will not affect the surface separability. An N-separating surface is called *N-minimal* if it does not contain any N-simple voxel, as shown in Figure 4.

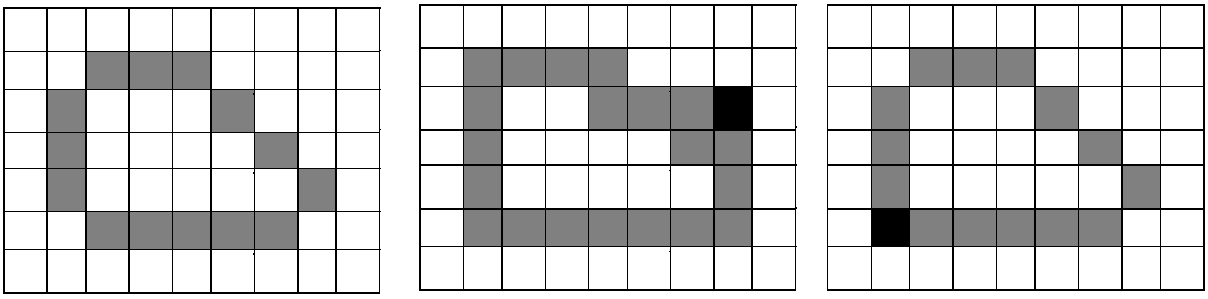


Figure 4. Examples of a 4-minimal curve (left), 8-simple point (center), and a 4-simple point (right).

Denote by S^d the set of all voxels that meet S. That is, if we define the extent of the unit voxels as having closed boundaries, each voxels that consist of a portion of S is in S^d. Denote by I^d and O^d the set of voxels that are wholly contained in I and O, respectively. One can show that S^d N-separates I^d and O^d, for any N. However, S^d is not always N-minimal, for any N. That is, although S^d is 'close' to the continuous surface, it contains too many voxels. By removing all N-simple voxels from S^d we can generate a surface $\tilde{S} \subseteq S^d$ that is both N-separating and N-minimal.

are said to be *N-connected in W* if there exists an N-path between them consisting only of voxels in *W*. It is easy to show that "connectedness in *W*" is an equivalence relation. The equivalence classes under this relation are the *connected components*. A *(closed) N-curve* is an N-path *P* that either contains a single voxel or that each of its voxels has exactly two N-adjacent voxels in *P*. An *open N-curve* is a N-curve with two exceptions called *endpoints*, each of which has only one N-adjacent voxel in *P*.

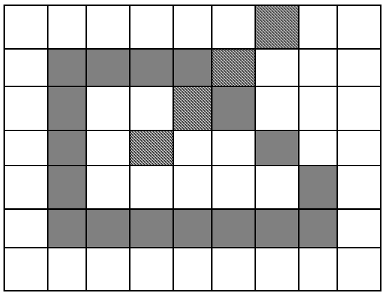


Figure 2. An 8-connected curve penetrating through another 8-curve.

In continuous space it is impossible to pass from the region enclosed by a curve to the region outside the curve without crossing the curve itself. In discrete space however, such paradox is possible. Figure 2 shows a 2D discrete 8-connected curve penetrating through another 8-curve without meeting it. To avoid such disagreement between the continuous and discrete spaces it has been the convention to define opposite types of connectivity for the white and black sets [14]. Opposite types in two dimensional space are 4 and 8, while in three dimensional space 6 is "opposite" to 26 or to 18. In 3D, however, the situation is much more complex because the connectivity of a surface does not fully characterizes its topology. The failure of the connectivity to properly characterize a surface is the main motivation behind the notion of separability that is the main concern of this work.

3. Topological and Geometrical Considerations

Let *S* be a continuous surface in R^3, we denote by $\tilde{S}$ a discrete representation of *S* as a set of voxels generated by some arbitrary procedure. There are three major requirements that $\tilde{S}$ should fulfill. First, we would like to preserve analogy between the continuous and discrete spaces by assuring that $\tilde{S}$ is penetrable if and only if *S* is. This requirement is called *separability* and it is formally defined and characterized in this section. In addition, $\tilde{S}$ should be the most accurate discrete representation of *S* according to some appropriate error metric. Finally, the voxelization should be minimal, that is, it should not contain voxels that their removal makes no difference in terms of separability. We now turn to describe each one of these requirements, namely separability, minimality, and accuracy, in more detail.

Let *A, B* and *C* be three disjoint sets of voxels. *A* is said to *N-separate B* and *C* if any *N*-path Π between a voxel in *B* and a voxel in *C* meets *A* (i.e, $\Pi \cap A \neq \emptyset$ Π. (See Figure 3).

cally an extension of the widely known 2D scan-line algorithm [16][8][17] where at each scan-line the third dimension is also interpolated. By selecting the correct orientation the algorithm guarantees that no gaps appear in the third dimension by employing numerical consideration.

2. Introduction to Discrete Space

A 3D discrete space Z^3 is a set of grid points in a 3D Euclidean space denoted by Σ. A 3D grid point is a zero dimension object defined by its Cartesian coordinate *(x,y,z)*. The *Voronoi neighborhood of grid point p* is the set of all points in the Euclidean space that are at least as close to *p* as to any other grid points. The Voronoi neighborhood of a 3D grid point is a unit cube around known also as a *voxel.*

The aggregate of all the voxels tessellates the 3D Euclidean space. A voxel is treated interchangeably in the literature as either a unit volume or a zero-dimensional point. In this paper, the term voxel is used only for a volumetric unit cube. Each voxel value is mapped into the set *{0,1}*: the voxels that are assigned the value “1" are called the “black" or “non-empty" voxels, while those that are assigned the value “0" are called the “white" or “empty" voxels.

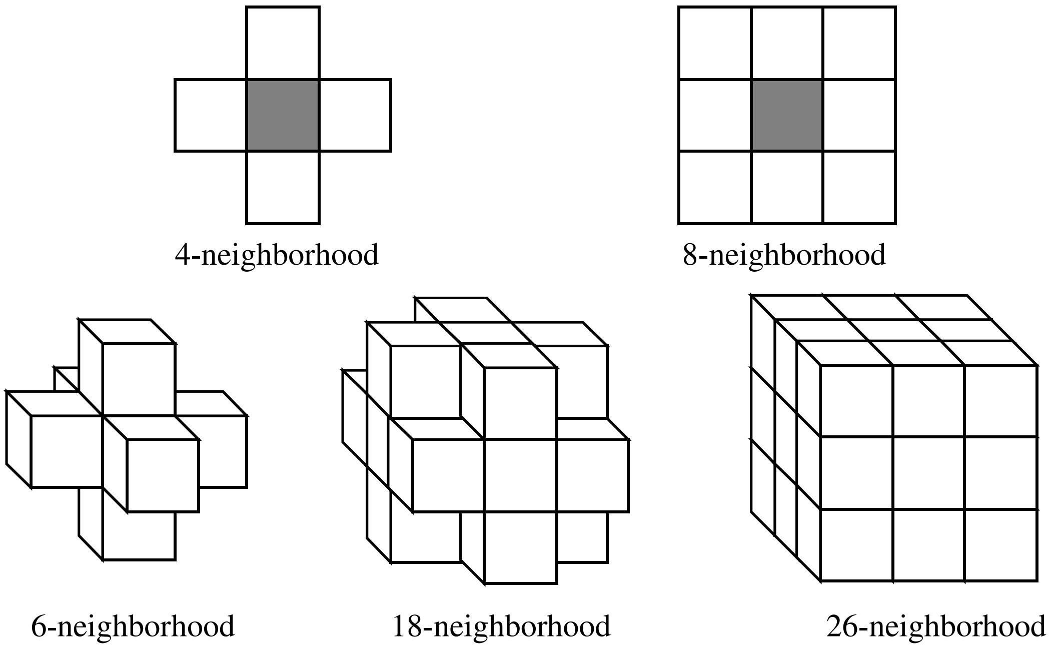


Figure 1. The set of 2D pixels that are N-adjacent to the dark pixel (top). The set of 3D voxels that are N-adjacent to the voxel at the center (bottom).

Two voxels are *26-adjacent* if they either share a vertex, an edge, or a face (see Figure 1). Every voxel has 26 such neighbors. Eight share a vertex (corner) with the center voxel, twelve share an edge, and six share a face. Accordingly, face-sharing voxels are defined as *6-adjacent*, and edge-sharing and face-sharing voxels are defined as *18-adjacent* (see Figure 1).

The prefix N is used to denote the adjacency relation, where $\forall \in \{6, 18, 26\}$. Let Σ be a set of voxels. A *N-path* is a sequence of voxels such that consecutive pairs are N-adjacent. Two voxels

Accurate Methods for the Voxelization of Planar Objects

Vassily Filippov and Roni Yagel

Department of Computer and Information Science
The Ohio State University

Abstract: *The process of generating discrete surfaces in volumetric representation, termed* voxelization*, is confronted with topological considerations as well as accuracy and efficiency requirements. We define the topological notions of separability and minimality and introduce a method for converting planar objects into voxel representation. Through geometric principles, our method provides topological consistency as well as numerical accuracy.*

1. Introduction

Planar objects are the most basic three-dimensional surfaces and they form the basis for wide range of other capabilities. Planar objects (e.g., polygons) are widely used as the basic surface primitive for 3D graphics, relying on the approximated representation of arbitrary surface as a mesh of polygonal patches.

In volumetric graphics, besides the obvious need for voxelizing planar objects for graphical purposes [9][20] and for intermixing them with sampled data [15], planar objects are used for other operations such as the extraction of an oblique cut-plane in which a plane is intersected with the volume, displaying all those voxels pierced by the plane. The identification of those voxels pierced by the plane is in essence a plane voxelization problem. Some block operations are performed on a sub-volume region bounded by planar clipping planes. The identification of those voxels belonging to the desired region involves the voxelization of its planar boundaries.

A voxelization algorithm for any geometric object faces several goals. Besides being efficient and accurate the algorithm must generate surfaces that are thick enough not to allow rays to penetrate to them (the separability requirement). However, we would not like surfaces to be 'too thick' but to contain only those voxels essential for separability (the minimality requirement).

Early work in scan-converting planar objects was concerned only with filling in 2D polygons or 2D projections of 3D polygons. A 2D polygon is defined as a closed sequence of line segments (edges). A scan-line algorithm for converting a polygon is widely known (e.g., [16][8]). Pavlidis [17] presented a similar scheme for scan-converting regions bounded by parabolic splines. Van Wyk [19] has proposed algorithms for scan-converting regions bounded by circular arcs. Hersch [10] has discussed such algorithms to partly degenerate shapes like character contours. A voxelization algorithm for planar polygons has been proposed in [11][12][13]. The algorithm is basi-