4 Open Questions

We have an exact characterization of closure of Gap-P and $\mathcal{Q}\#$P under integer division by polynomially-bounded functions. However, we do not know of a similar characterization of closure of #P.

Is there a direct proof of PH $\subseteq$ PP, if #P or Gap-P is closed under integer division by $s(x)$? The known proofs, such as in Corollary 3.2, do use Toda's result. Note that the proof of Corollary 3.6 does not require Toda's result as Proposition 3.3 and Theorems 3.4 and 3.5 can be shown with straightforward arguments.

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References

[FFK91] S. Fenner, L. Fortnow, and S. Kurtz. Gap-definable counting classes. In *Proceedings of 6th Structure in Complexity Theory Conference*, pages 30–42. IEEE Computer Society Press, 1991.

[Gup91] S. Gupta. The power of witness reduction. In *Proceedings of 6th Annual Structure in Complexity Theory Conference*, pages 43–59. IEEE Computer Society Press, 1991.

[OH91] M. Ogiwara and L.A. Hemachandra. A complexity theory for feasible closure properties. In *Proceedings of 6th Annual Structure in Complexity Theory Conference*, pages 16–29. IEEE Computer Society Press, 1991.

[Reg82] K. Regan. Enumeration problems. 1982.

[Sto77] L. Stockmeyer. The polynomial-time hierarchy. *Theor. Comput. Sci.*, 3:1–22, 1977.

[TO90] S. Toda and M. Ogiwara. Counting classes are at least as hard as the the polynomial-time hierarchy. In *Proceedings of 6th Structure in Complexity Theory Conference*, pages 2–12. IEEE Computer Society Press, 1990.

[Tod89] S. Toda. On the computational power of PP and $\oplus$P. In *Proc. 30th IEEE Symp. on Foundations of Computer Science*, pages 514–519, 1989.

[Val79] L. Valiant. The complexity of computing the permanent. *SIAM J. Comput.*, 8:189–201, 1979.

[Wag86] K. Wagner. The complexity of combinatorial problems with succint input representation. *Acta Informatica*, 23:325–356, 1986.

Proposition 3.3 *Let $s \in$ PF be a function which maps strings to natural numbers such that, for all strings x, $2 \leq s(x) \leq |x|^k + k$, for some fixed k. $\mathcal{Q}\#$P is closed under integer division by $s(x)$ iff $\mathcal{Q}\#$P is closed under integer division by an arbitrary $\mathcal{Q}\#$P function.*

Proof. If $\mathcal{Q}\#$P is closed under integer division by an arbitrary $\mathcal{Q}\#$P function, then $\mathcal{Q}\#$P is clearly closed under integer division by $s(x)$.

Let $\mathcal{Q}\#$P be closed under integer division by $s(x)$. Let $\frac{f_1}{f_2}$, $\frac{g_1}{g_2}$, $f_2, g_1, g_2 \neq 0$ be two $\mathcal{Q}\#$P functions where $f_1, f_2, g_1, g_2 \in$ Gap-P. Then, $\frac{f_1}{f_2}/\frac{g_1}{g_2} = \frac{f_1 g_2}{f_2 g_1} = \frac{f}{g}$, where $f, g \in$ Gap-P, as Gap-P is closed under multiplication. Thus it is sufficient to show that $\lfloor \frac{f}{g} \rfloor \in \mathcal{Q}\#$P.

Consider $\frac{s*f}{g}$ which is in $\mathcal{Q}\#$P, as Gap-P is closed under multiplication. By our hypothesis, there exists a $h \in \mathcal{Q}\#$P such that for all x, $h(x) = \lfloor \frac{s(x)f(x)}{g(x)}/s(x) \rfloor = \lfloor \frac{f(x)}{g(x)} \rfloor$, for all x. □

The following two theorems were proved in [Gup91].

Theorem 3.4 *$\mathcal{Q}\#$P is closed under integer division iff* PP = C$_=$P.

Theorem 3.5 *If* PP = C$_=$P *then* $\underbrace{\exists \cdot \forall \cdots}_{k>0}$PP = PP.

Corollary 3.6 *(to Proposition 3.3, Theorem 3.4 and Theorem 3.5) Let $s \in$ PF be a function which maps strings to natural numbers such that, for all strings x, $2 \leq s(x) \leq |x|^k + k$, for some fixed k. If $\mathcal{Q}\#$P is closed under integer division by $s(x)$ then,* PH $\subseteq$ PP.

Proof. If $\mathcal{Q}\#$P is closed under integer division by $s(x)$ then it is closed under integer division, which implies PP = C$_=$P. Then, PH = $\cup_{k \geq 0} \underbrace{\exists \cdot \forall \cdot \exists \cdots}_{k}$P $\subseteq \cup_{k \geq 0} \underbrace{\exists \cdot \forall \cdot \exists \cdots}_{k}$PP $\subseteq$ PP. □

The results of Theorem 3.1, Corollary 2.7, Corollary 3.2 and Corollary 3.6 can be summarized in the following corollary.

Corollary 3.7 *Let $s \in$ PF be a function which maps strings to natural numbers such that, for all strings x, $2 \leq s(x) \leq |x|^k + k$, for some fixed k. If any of* #P *or* Gap-P *or $\mathcal{Q}\#$P are closed under integer division by $s(x)$, then* PH $\subseteq$ PP.

1. Gap-P *is closed under integer division by* $s(x)$.

2. $\mathrm{SPP} = \mathrm{MOD_sP}$.

3. $\#\mathrm{P}^{\mathrm{MOD_sP}} \subseteq$ Gap-P.

Proof. We show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$[1 \Rightarrow 2]$ It is easy to see that $\mathrm{SPP} \subseteq \mathcal{Z}\mathrm{MOD_sP} = \mathrm{MOD_sP}$ from the definitions and Corollary 2.6. Thus we need to show that $\mathrm{MOD_sP} \subseteq \mathrm{SPP}$. Let $L \in \mathrm{MOD_sP}$; that is, there exists $f \in \#\mathrm{P}$ such that, $x \in L \Leftrightarrow f(x) \not\equiv 0 (\mathrm{mod}\ s(x))$. Define $g(x) = f(x) - s(x)\lfloor \frac{f(x)}{s(x)} \rfloor$. As Gap-P is closed under multiplication and subtraction, and by our hypothesis Gap-P is closed under integer division by $s(x)$, therefore $g \in$ Gap-P. Then, $0 \leq f(x) < s(x) \leq |x|^k + k$ and $x \in L \Leftrightarrow g(x) \neq 0$. Thus, $L \in$ Gap-Few = SPP from part 7 of Proposition 2.4.

$[2 \Rightarrow 3]$ $\#\mathrm{P}^{\mathrm{MOD_sP}} \subseteq \text{Gap-P}^{\mathrm{MOD_sP}} \subseteq \text{Gap-P}^{\mathrm{SPP}} \subseteq$ Gap-P follows from our hypothesis and part 1 and 5 of Proposition 2.4.

$[3 \Rightarrow 1]$ Given $f \in$ Gap-P we need to show that $\lfloor \frac{f}{s} \rfloor \in$ Gap-P. By propostion 2.5, $f(x) = g(x) - s(x)2^{q(|x|)}$, where $g \in \#\mathrm{P}$ and q is some polynomial. Let $g(x) = \mathrm{Acc}(M, x)$ for some NTM M. Define $B = \{\langle M, x, \tau \rangle \mid M$ on input x has τ as the ith accepting computation path lexicographically and $i = 0 (\mathrm{mod}\ s(x))\}$. It is easy to see that $\bar{B} \in \mathrm{MOD_sP}$. Consider the following oracle machine $\hat{M}$ which queries the set B. $\hat{M}$ simulates the NTM M on input x. If M rejects then so does $\hat{M}$. If M accepts on a computation path τ, $\hat{M}$ queries the set B with $\langle M, x, \tau \rangle$. $\hat{M}$ accepts if and only if oracle answers "no".

It is easy to see that $\mathrm{Acc}(\hat{M}, x) = \lfloor \frac{g(x)}{s(x)} \rfloor$. Thus, $\lfloor \frac{g}{s} \rfloor \in \#\mathrm{P}^{\mathrm{MOD_sP}} \subseteq$ Gap-P under our hypothesis. But, $\lfloor \frac{f}{s} \rfloor = \lfloor \frac{g}{s} \rfloor - 2^{q(|x|)}$ which is in Gap-P. □

Corollary 3.2 *Let* $s \in \mathrm{PF}$ *be a function which maps strings to natural numbers such that, for all strings* x, $2 \leq s(x) \leq |x|^k + k$, *for some fixed* k. *If* Gap-P *is closed under integer division by* $s(x)$ *then* $\mathrm{PH} \subseteq \mathrm{PP}$.

Proof. $\mathrm{PH} \subseteq \mathrm{BP} \cdot \mathrm{MOD_sP} \subseteq \mathrm{BP} \cdot \mathrm{SPP} \subseteq \mathrm{PP}$ using part 4 and 6 of Proposition 2.4. □

Next, we show that $\mathcal{O}\#\mathrm{P}$ has a similar characterization for closure under integer division by polynomially-bounded functions.

8. *[FFK91] Let $f \in$ Gap-P. Then, there exists a $g \in \#\mathrm{P}$, and a polynomial q such that, for all x, $f(x) = g(x) - 2^{q(|x|)}$.*

Our proofs will also use the following easily provable results.

Proposition 2.5 *Let $s \in$ PF be a function which maps strings to non-zero natural numbers. Let $f \in$ Gap-P. Then, there exists a $g \in \#\mathrm{P}$, and a polynomial q such that, for all x, $f(x) = g(x) - s(x)2^{q(|x|)}$.*

Proof. From part 8 of Proposition 2.4 $f(x) = g'(x) - 2^{q(|x|)}$, where $g' \in \#\mathrm{P}$ and q is a polynomial. Then $g(x) = g'(x) + 2^{q(|x|)}(s(x) - 1)$ satisfies the requirements of the theorem. Note that $s(x) - 1$ is in PF and hence $g(x) \in \#\mathrm{P}$. □

Corollary 2.6 *Let $s \in$ PF be a function which maps strings to non-zero natural numbers. Then, $\mathcal{Z}\mathrm{MOD_sP} = \mathrm{MOD_sP}$.*

Proof. $\mathrm{MOD_sP} \subseteq \mathcal{Z}\mathrm{MOD_sP}$ follows from the fact that $\#\mathrm{P} \subseteq$ Gap-P. Let $L \in \mathcal{Z}\mathrm{MOD_sP}$ be witnessed by $f \in$ Gap-P. Let $g \in \#\mathrm{P}$ and polynomial q be as given by proposition 2.5 such that, $f(x) = g(x) - s(x)2^{q(|x|)}$. Then, $f(x) = g(x) (\mathrm{mod\ s(x)})$. □

Corollary 2.7 *Let $s \in$ PF be a function which maps strings to non-zero natural numbers. If $\#\mathrm{P}$ is closed under integer division by $s(x)$, then* Gap-P *is closed under integer division by $s(x)$.*

Proof. Let $f \in$ Gap-P. By Proposition 2.5 $f(x) = g(x) - s(x)2^{q(|x|)}$, where $g \in \#\mathrm{P}$ and q is some polynomial. Then, $\lfloor \frac{f(x)}{s(x)} \rfloor = \lfloor \frac{g(x)}{s(x)} \rfloor - 2^{q(|x|)}$. Thus closure of g under integer division by $s(x)$ implies closure of f under integer division by $s(x)$. □

3 Main Results

Now we give an exact characterization of the closure of Gap-P under integer division by polynomial bounded functions.

Theorem 3.1 *Let $s \in$ PF be a function which maps strings to natural numbers such that, for all strings x, $2 \leq s(x) \leq |x|^k + k$, for some fixed k. Then, the following statements are equivalent.*

5. *A language* $L \in \mathrm{SPP}$ *iff there exists* $f \in$ Gap-P *such that, for all* x, $x \in L \Rightarrow f(x) = 1$ *and* $x \notin L \Rightarrow f(x) = 0$.

6. *A language* $L \in$ Gap-Few *iff there exists* $f \in$ Gap-P, *a polynomial* q, *and a polynomial-time computable predicate* $A(x, y)$ *such that, for all* x, $0 \leq f(x) \leq q(|x|)$ *and* $x \in L \Leftrightarrow A(x, f(x))$.

7. *A language* $L \in \mathrm{C_{=}P}$ *iff there exists a function* $f \in \mathrm{PF}$, $f : \Sigma^* \longmapsto \mathcal{N}$, *and* $g \in \#\mathrm{P}$ *such that, for all* x, $x \in L \Leftrightarrow g(x) = f(x)$.

8. *A language* $L \in \mathrm{PP}$ *iff there exists a function* $f \in \mathrm{PF}$, $f : \Sigma^* \longmapsto \mathcal{N}$, *and* $g \in \#\mathrm{P}$ *such that, for all* x, $x \in L \Leftrightarrow g(x) \geq f(x)$.

9. *Let* $s \in \mathrm{PF}$ *be a function which maps strings to non-zero natural numbers. Then* $L \in \mathrm{MOD_sP}$ *if there exists a* $f \in \#\mathrm{P}$ *such that* $x \in L \Leftrightarrow f(x) \not\equiv 0 (\mathrm{mod}\ s(x))$.

10. *Let* $s \in \mathrm{PF}$ *be a function which maps strings to non-zero natural numbers. Then* $L \in \mathcal{Z}\mathrm{MOD_sP}$ *if there exists a* $f \in$ Gap-P *such that* $x \in L \Leftrightarrow f(x) \not\equiv 0 (\mathrm{mod}\ s(x))$.

11. $\oplus\mathrm{P} = \mathrm{MOD_2P}$.

Our proofs use the following proposition which describes some of the known results.

Proposition 2.4

1. *[FFK91, Gup91]* $\#\mathrm{P} \subseteq$ Gap-P $\subseteq \mathcal{Q}\#\mathrm{P}$.

2. *[FFK91, Gup91]* Gap-P *is closed under addition, subtraction and multiplication (on Z).*

3. *[Gup91]* $\mathcal{Q}\#\mathrm{P}$ *is closed under addition, subtraction, multiplication and division(on Q).*

4. *[Gup91]* $\mathrm{BP} \cdot \mathrm{SPP} \subseteq \mathrm{PP}$.

5. *[FFK91]* Gap-$\mathrm{P}^{\mathrm{SPP}} \subseteq$ Gap-P, *or,* SPP *is low for* Gap-P.

6. *[TO90] Let* $s \in \mathrm{PF}$ *be a function which maps strings to natural numbers such that, for all strings* x, $s(x) \geq 2$, *for some fixed* k. *Then,* $\mathrm{PH} \subseteq \mathrm{BP} \cdot \mathrm{MOD_sP}$.

7. *[FFK91]* Gap-Few = SPP.

We define the following functions on nondeterministic Turing machines.

Definition 2.1 *Given a nondeterministic machine N, let* $\mathrm{Acc}(N,x)$ = *number of accepting paths of N on input x, and let* $\mathrm{Rej}(N,x)$ = *number of rejecting paths of N on input x. Then, let* $\mathrm{AR}(N,x) = \mathrm{Acc}(N,x) - \mathrm{Rej}(N,x)$.

Next, we define the function classes #P, Gap-P and $\mathcal{Q}$#P.

Definition 2.2

1. *[Val79]A function $f : \Sigma^* \longmapsto \mathcal{N}$ is in* #P *iff there exists a polynomial-time bounded nondeterministic machine M such that for all x, $f(x) = \mathrm{Acc}(M,x)$.*

2. *[FFK91, Gup91] A function $f : \Sigma^* \longmapsto \mathcal{Z}$ is in* Gap-P *iff there exists a polynomial-time bounded nondeterministic machine M such that for all x, $f(x) = \mathrm{AR}(M,x)$. Equivalently, $f \in$ Gap-P iff for all x, $f(x) = g(x) - h(x)$, for some $g, h \in$ #P.*

3. *[Gup91] A function $f : \Sigma^* \longmapsto \mathcal{Q}$ is in $\mathcal{Q}$*#P *iff there exists a pair of polynomial-time bounded nondeterministic machines (M^n, M^d) such that for all x, $\mathrm{AR}(M^d,x) \neq 0$ and $f(x) = \frac{AR(M^n,x)}{AR(M^d,x)}$. Equivalently, $f \in \mathcal{Q}$*#P *iff for all x, $f(x) = \frac{g(x)}{h(x)}$ and $h(x) \neq 0$, for some $g, h \in$ Gap-P.*

Next, we define the various quantifiers and language classes used in this paper.

Definition 2.3

1. *A language $L \in \mathrm{BP} \cdot K$ iff there exists a polynomial p and a set $A \in K$ such that, Prob$(x \in L \Leftrightarrow \langle x,y\rangle \in A) \geq \frac{3}{4}$, where y is chosen uniformly from $\{0,1\}^{p(|x|)}$.*

2. *A language $L \in \exists \cdot K$ iff there exists a polynomial p and a set $A \in K$ such that, $x \in L \Leftrightarrow \|\{y \mid |y| = p(|x|), \langle x,y\rangle \in A\}\| \geq 1$.*

3. *A language $L \in \forall \cdot K$ iff there exists a polynomial p and a set $A \in K$ such that, $x \in L \Leftrightarrow \|\{y \mid |y| = p(|x|), \langle x,y\rangle \in A\}\| = 2^{p(|x|)}$.*

4. *A language $L \in$ UP iff there exists $f \in$ #P such that, for all x, $x \in L \Rightarrow f(x) = 1$ and $x \notin L \Rightarrow f(x) = 0$.*

The closure of Gap-P under integer division by 2 remains open. In this paper we give an exact characterization of this closure. Our main theorem is as follows:

Theorem 3.1 Let $s \in \mathrm{PF}$ be a function which maps strings to natural numbers such that, for all strings x, $2 \leq s(x) \leq |x|^k + k$, for some fixed k. Then, the following statements are equivalent.

1. Gap-P is closed under integer division by $s(x)$.
2. $\mathrm{SPP} = \mathrm{MOD_sP}$.
3. $\#\mathrm{P}^{\mathrm{MOD_sP}} \subseteq$ Gap-P

In particular when $s(x) = 2$, for all x, we have an exact characterization of the closure of Gap-P under division by 2, $\mathrm{SPP} = \mathrm{MOD_2P} = \oplus\mathrm{P}$. Secondly, note that #P is closed under integer division by $s(x)$ implies that Gap-P is closed under integer division by $s(x)$ (This is shown in Corollary 2.7). Thus, closure of Gap-P under integer division by $s(x)$, a seemingly weaker hypothesis than #P is closed under integer division by $s(x)$, has the same interesting consequences.

Finally, we show in Proposition 3.3 that $\mathcal{Q}\#\mathrm{P}$ is closed under integer division by a polynomially-bounded function iff $\mathcal{Q}\#\mathrm{P}$ is closed under integer division by an arbitrary $\mathcal{Q}\#\mathrm{P}$ function. This, together with the results shown in [Gup91], gives the following corollary: if $\mathcal{Q}\#\mathrm{P}$ is closed under integer division by polynomially-bounded functions, then PH $\subseteq$ PP.

Thus, the closure of #P, Gap-P or $\mathcal{Q}\#\mathrm{P}$ under integer division by polynomially-bounded functions has the following interesting consequences.

Corollary 3.7 Let $s \in \mathrm{PF}$ be a function which maps strings to natural numbers such that, for all strings x, $2 \leq s(x) \leq |x|^k + k$, for some fixed k. If any of #P or Gap-P or $\mathcal{Q}\#\mathrm{P}$ are closed under integer division by $s(x)$, then PH $\subseteq$ PP.

2 Preliminaries

Our model of computation is polynomial-time bounded nondeterministic Turing machines(NTMs). Each computation path of the machine always ends in either accept or reject. All languages are over the fixed alphabet $\Sigma = \{0, 1\}$. The set of natural numbers, the set of integers, and the set of rational numbers, are denoted by $\mathcal{N}$, $\mathcal{Z}$ and $\mathcal{Q}$, respectively. We denote by PF the class of polynomial-time computable functions. We use $\langle \cdot, \cdot \rangle$ to denote some standard polynomial-time computable pairing function. Furthermore, we use $\langle \cdot, \cdot, \cdot \rangle$ to denote $\langle \langle \cdot, \cdot \rangle, \cdot \rangle$.

rational numbers). It is not known if Gap-P or $\mathcal{Q}\#$P are closed under integer division or proper subtraction.

Gupta, in [Gup91], showed that the results of Ogiwara and Hemachandra have analogues in Gap-P and $\mathcal{Q}\#$P. All the classes mentioned below are defined in the next section.

Theorem 1.2 *[Gup91] The following are equivalent.*

1. Gap-P *is closed under integer division.*
2. Gap-P *is closed under proper subtraction.*
3. Gap-P *is closed under every* PF*-function.*
4. CH = PP = SPP[4].

Theorem 1.3 *[Gup91] The following are equivalent.*

1. $\mathcal{Q}\#$P *is closed under integer division.*
2. $\mathcal{Q}\#$P *is closed under proper subtraction.*
3. $\mathcal{Q}\#$P *is closed under every* PF*-function.*
4. PP = $\mathrm{C}_{=}$P.

Thus, if Gap-P or $\mathcal{Q}\#$P are closed under integer division or proper subtraction, we have interesting collapses.

All the above proofs required closure under integer division by $2^{p(|x|)}$ for some polynomial p. Ogiwara and Hemachandra also considered a seemingly simpler operation, closure of #P under integer division by 2. They showed that if #P is closed under integer division by 2, then SPP = $\oplus$P. It is easy to see that SPP $\subseteq \oplus$P, though it is not known if this containment is proper. However, using Toda's result [Tod89] that PH $\subseteq$ BP $\cdot \oplus$P, $\oplus$P = SPP implies that PH $\subseteq$ BP $\cdot$ SPP $\subseteq$ PP[Gup91]. Thus, the question of whether $\oplus$P = SPP is of considerable interest. We also note that it is not known if $\oplus$P = SPP implies that #P is closed under integer division by 2.

[4]SPP is denoted as $\mathcal{Z}$UP in [Gup91].

Keywords: Computational Complexity, Function classes, Closure properties.

1 Introduction

One of the most important issues of complexity theory has been the closure of language classes under various operations such as union and intersection, and the closure of function classes under functions such as addition and multiplication. For example, consider the well-known result that #P[Val79] is closed under addition[Reg82]. This means that given two functions $f, g \in \#\mathrm{P}$, there exists an $h \in \#\mathrm{P}$ such that for all x, $h(x) = f(x) + g(x)$. It is known that #P is closed under multiplication as well, but the closure of #P under integer division[1] and proper subtraction[2] is not known. (Note that the range of functions in #P is the set of natural numbers, and hence we cannot talk about closure of #P under normal subtraction and division.)

Ogiwara and Hemachandra [OH91] broke new ground by showing that closure of #P under proper subtraction or integer division would have surprising and unexpected consequences:

Theorem 1.1 *[OH91] The following are equivalent.*

1. #P *is closed under proper subtraction.*
2. #P *is closed under integer division.*
3. #P *is closed under every* PF*-function.*
4. CH = PH = PP = UP.

CH denotes the counting hierarchy[Wag86] and PH denotes the polynomial-time hierarchy[Sto77]. Thus, both the counting hierarchy and the polynomial-time hierarchy collapse to UP iff #P is closed under proper subtraction or integer division.

The reason for the collapse of the counting hierarchy to UP under the closure of #P under proper subtraction or integer division is not clear from the above result, due to the restricted range of #P functions. To study the reason for this collapse, Gupta[Gup91] introduced Gap-P[3] and $\mathcal{Q}\#\mathrm{P}$, which are natural extensions of #P. Gap-P is the closure of #P under subtraction (over the set of integers), and $\mathcal{Q}\#\mathrm{P}$ is the closure of Gap-P under division (over the set of

[1] Integer division is defined as $a \div b = \lfloor \frac{a}{b} \rfloor$. Equivalently, we can define it as $\lceil \frac{a}{b} \rceil$.
[2] Proper subtraction is defined as $a \ominus b = \max\{a - b, 0\}$.
[3] The notation Gap-P was first introduced in [FFK91]; the same class is denoted $\mathcal{Z}\#\mathrm{P}$ in [Gup91].

On the Closure of Certain Function Classes under Integer Division by Polynomially-Bounded Functions [1] [2]

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[2]Part of this work was done while the author was a visitor in the Department of Computer Science at New Mexico State University.