Heaps and Heapsort

Heaps

- A ***heap*** is a ***`binary tree of T`*** that satisfies two properties:
  - Global shape property: it is a ***complete binary tree***
  - Local ordering property: the label in each node is “smaller than or equal to” the label in each of its child nodes

Heaps

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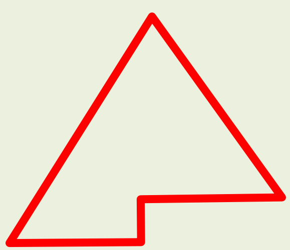
  - Global shape property: it is a ***complete***

in each

” the label in

A ***complete*** binary tree is one in which all levels are “full” except possibly the bottom level, with any nodes on the bottom level as far left as possible.



Heaps

Also in the picture is (as with BSTs, sorting, etc.) a total preorder that makes this notion precise.

e ***of*** *T* that

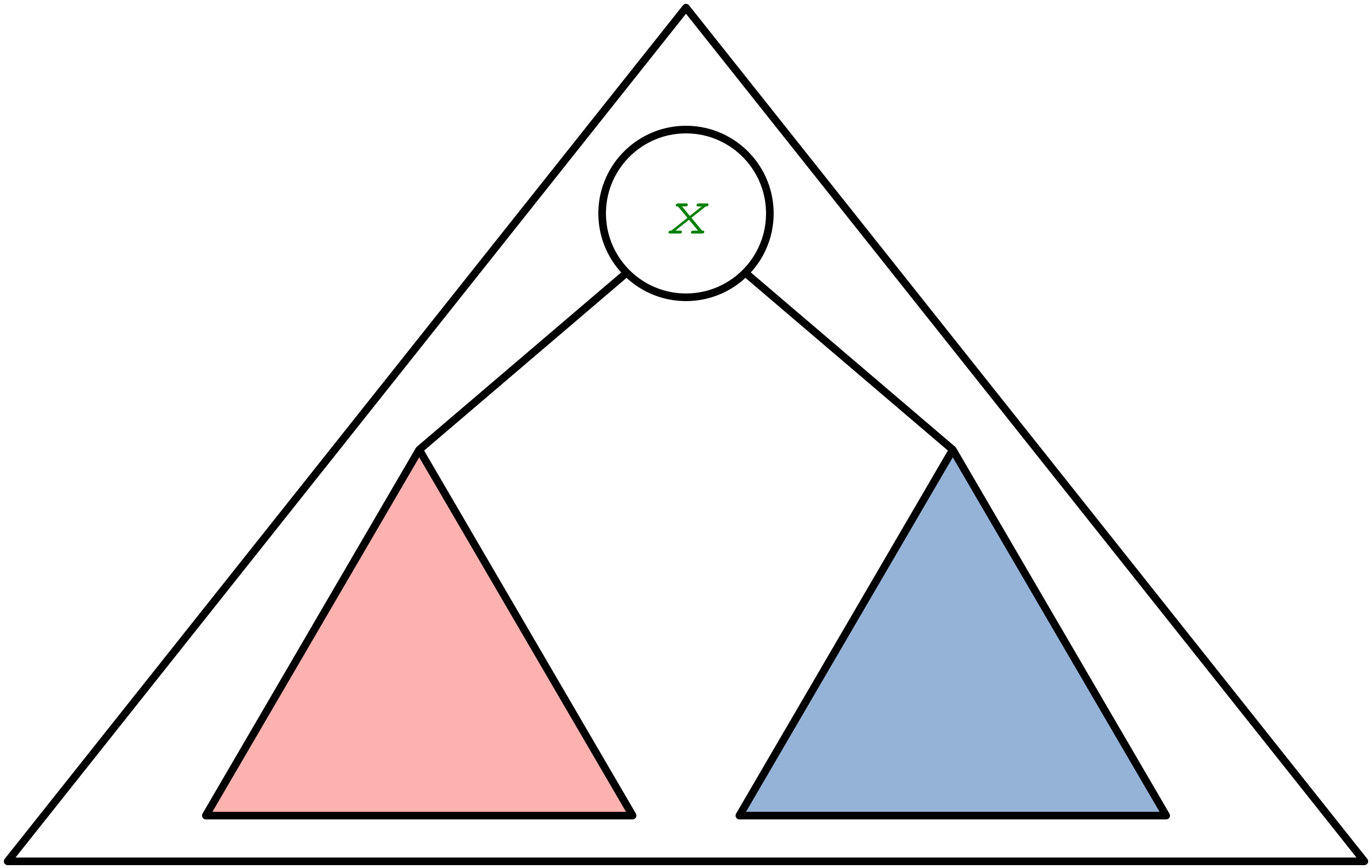
a ***complete binary tree***

– Local ordering property: the label in each node is “smaller than or equal to” the label in each of its child nodes

Simplification

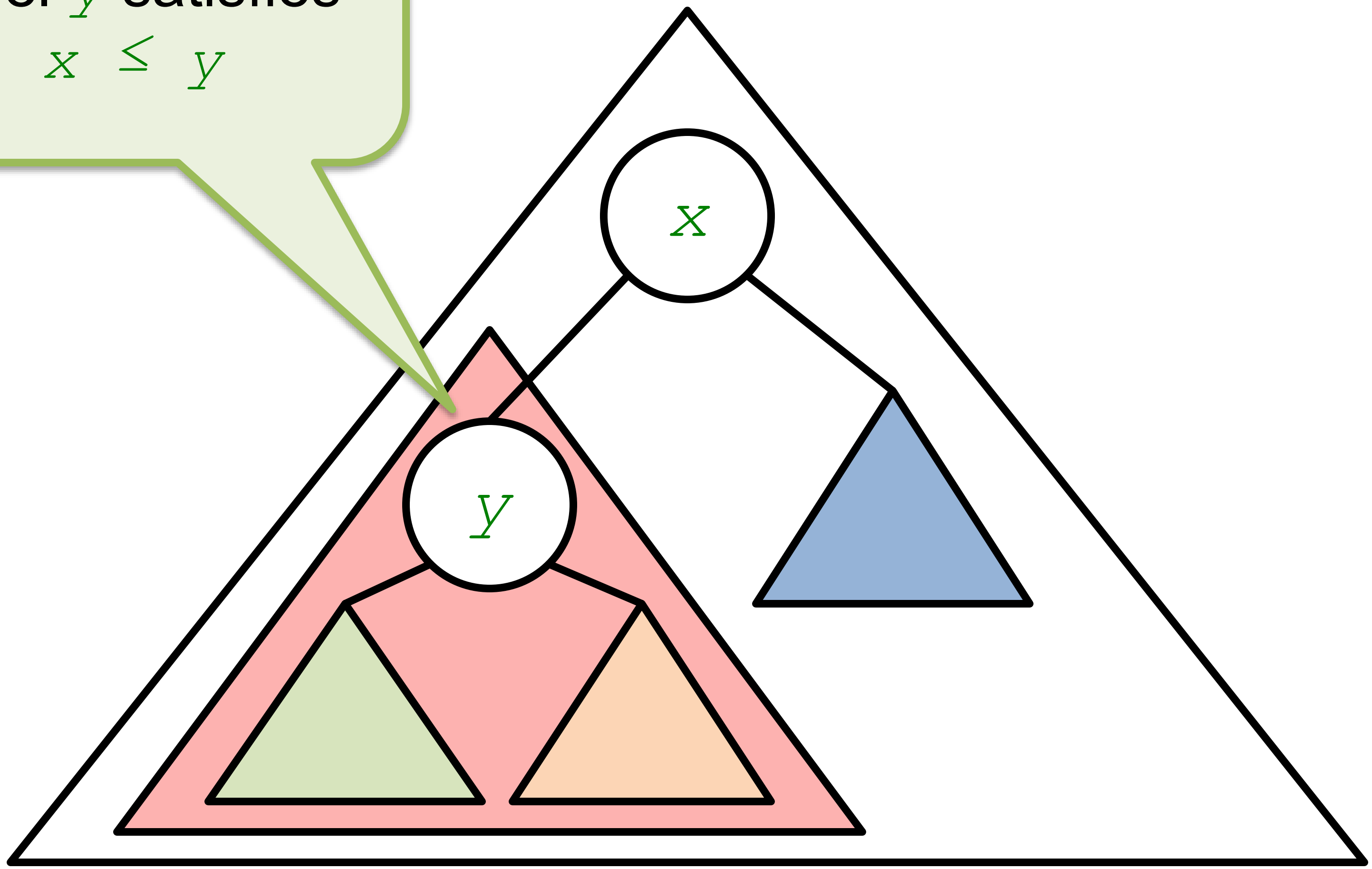
- For simplicity in the following illustrations, we use only one kind of example:
  - T = ***integer***
  - The ordering is $\leq$
- Because heaps are used in sorting, where duplicate values may be involved, we allow that multiple nodes in a heap may have the same labels (i.e., we will *not* assume that the labels are unique)

The Big Picture



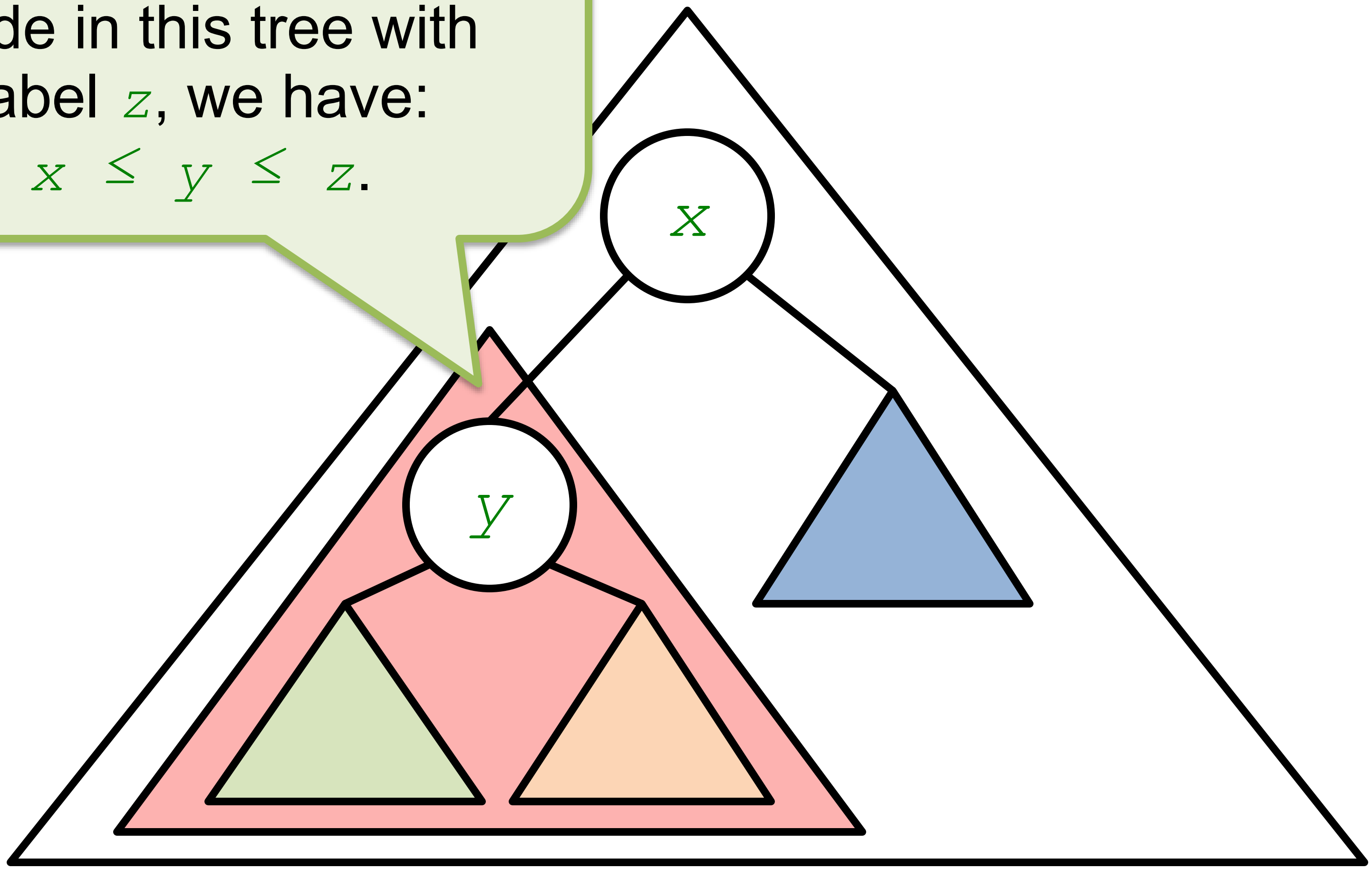
he Big Picture

This tree's root label y satisfies
$x \leq y$

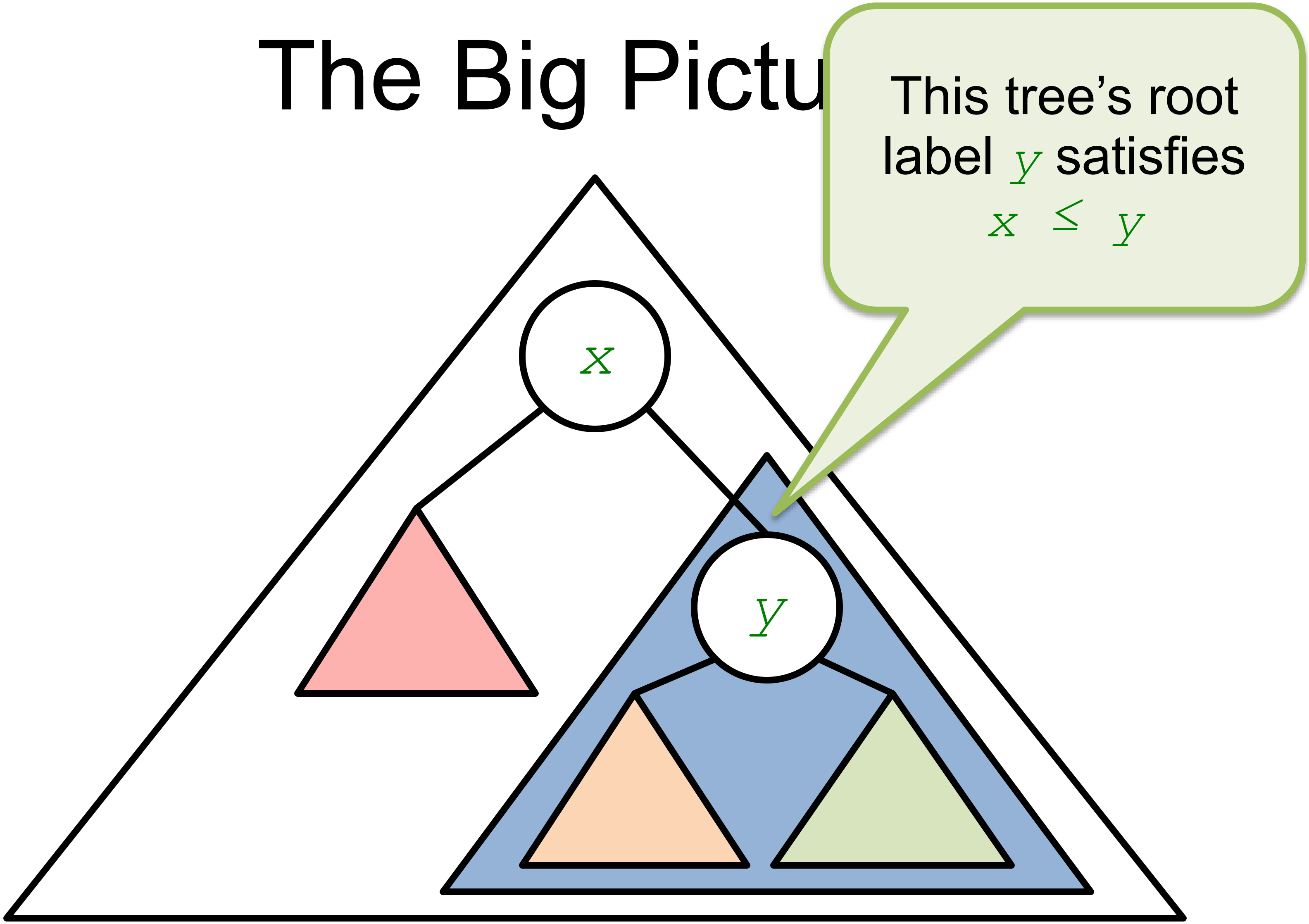


ig Picture

Observe: This tree is also a heap; and for each node in this tree with label z, we have: $x \leq y \leq z$.



The Big Pictu



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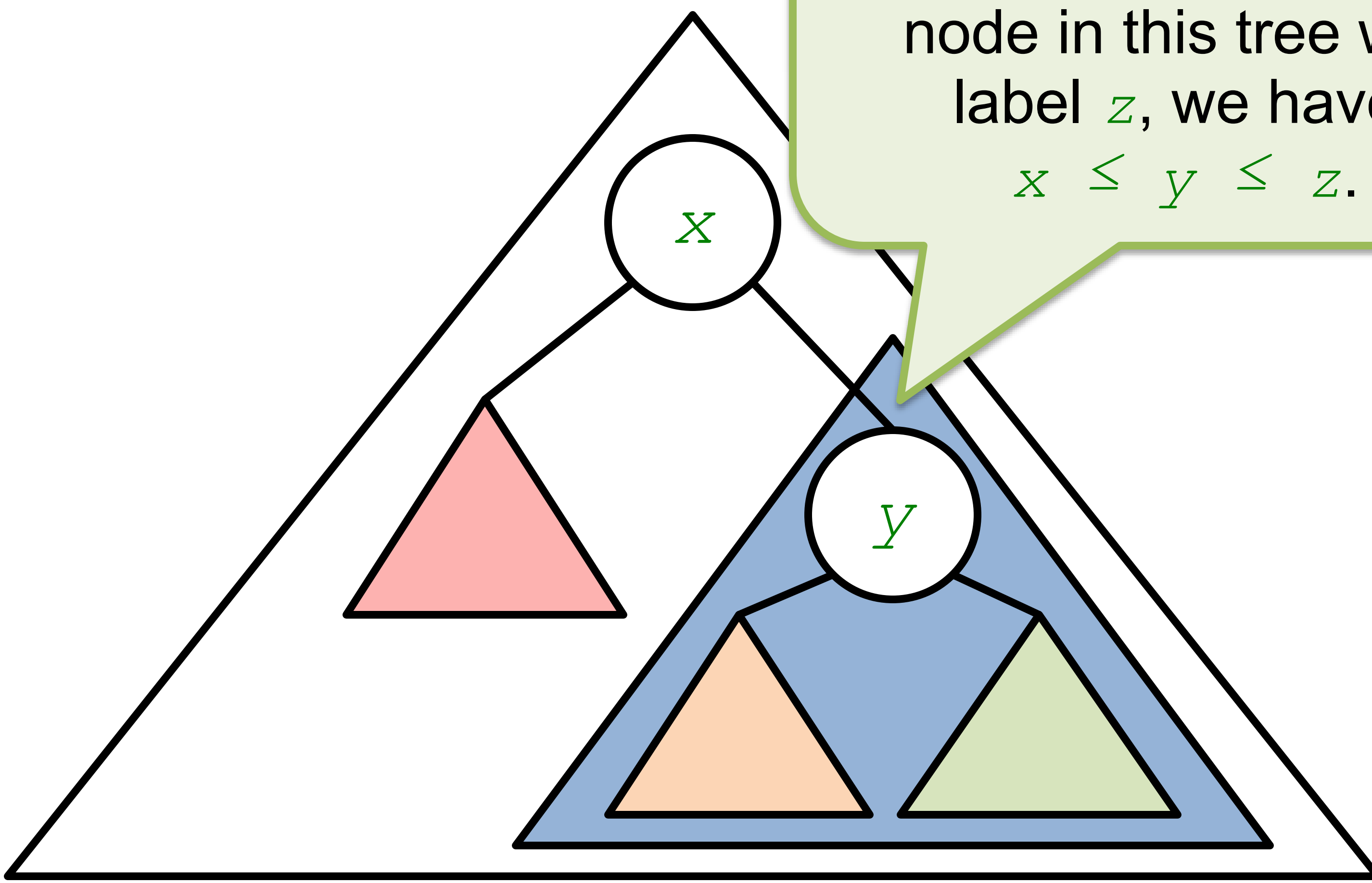
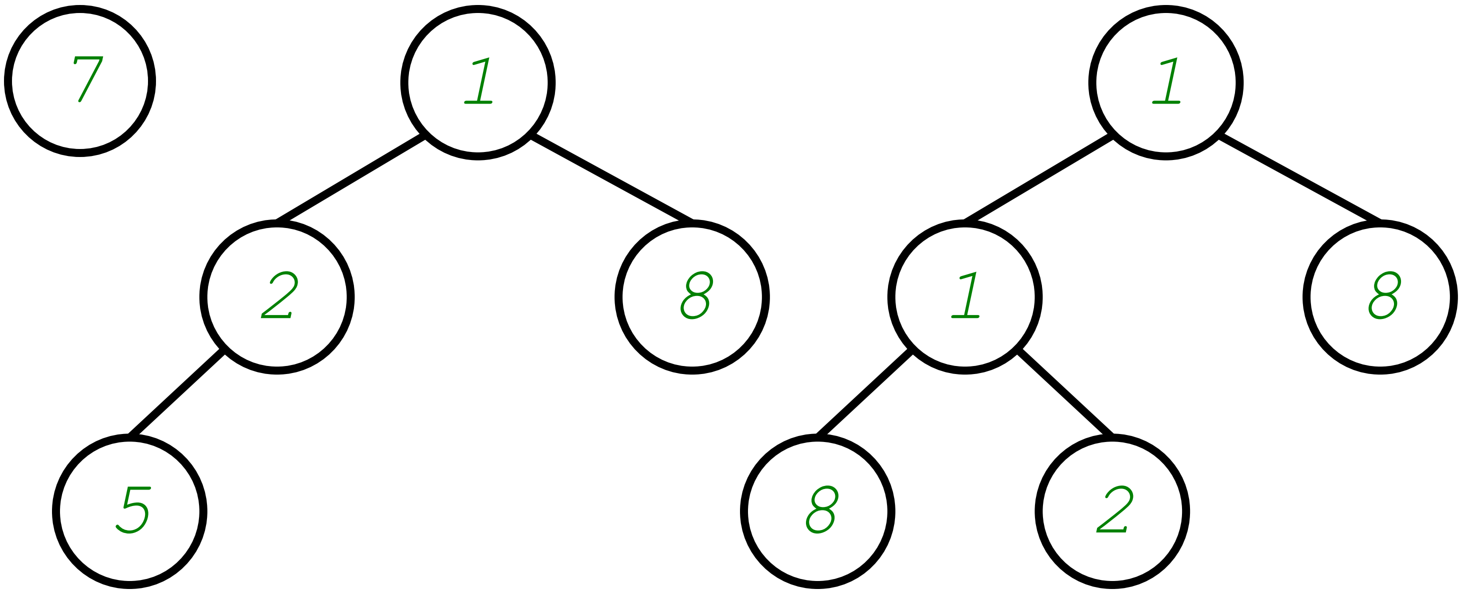
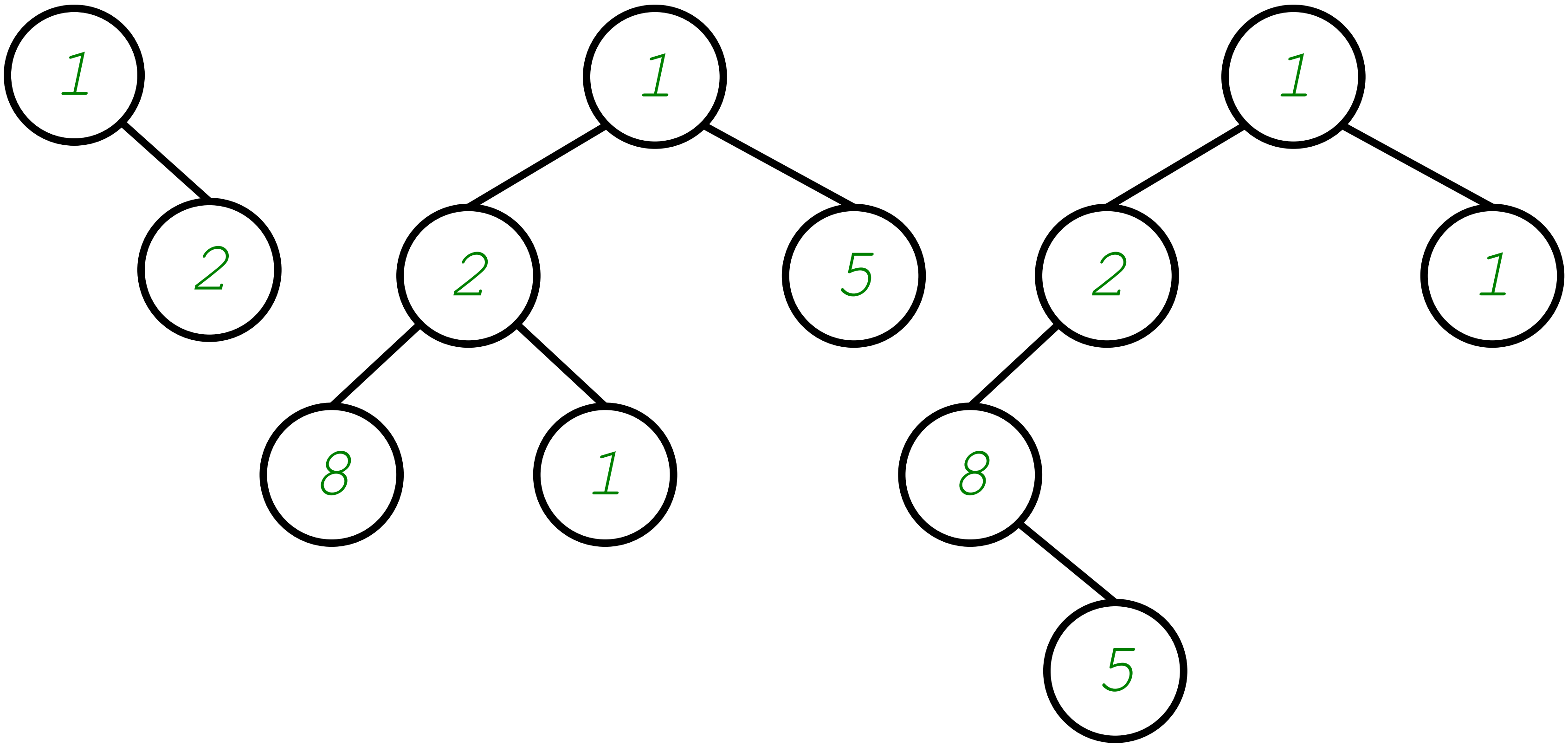


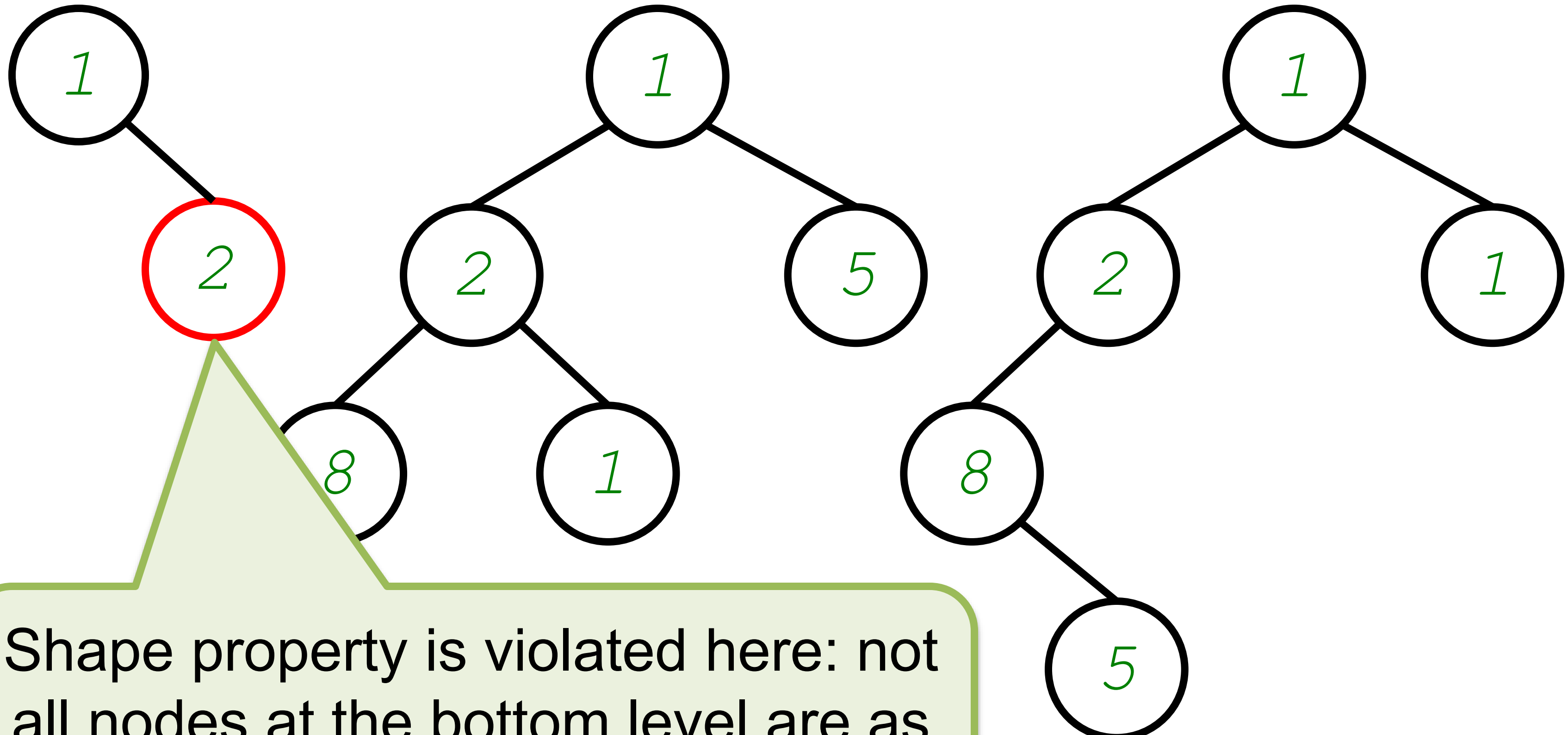
Examples of Heaps

- 

Non-Examples of Heaps

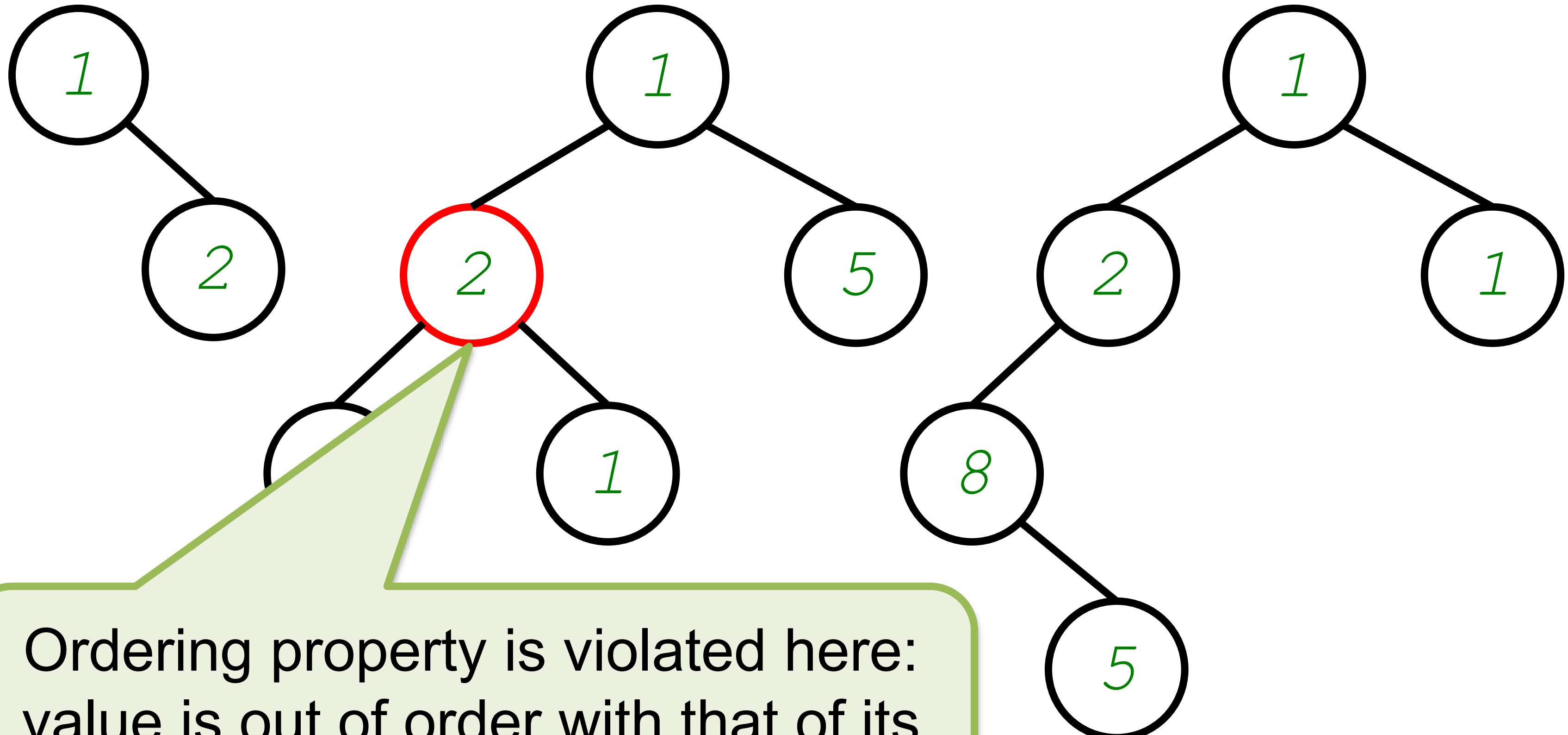


Non-Examples of Heaps



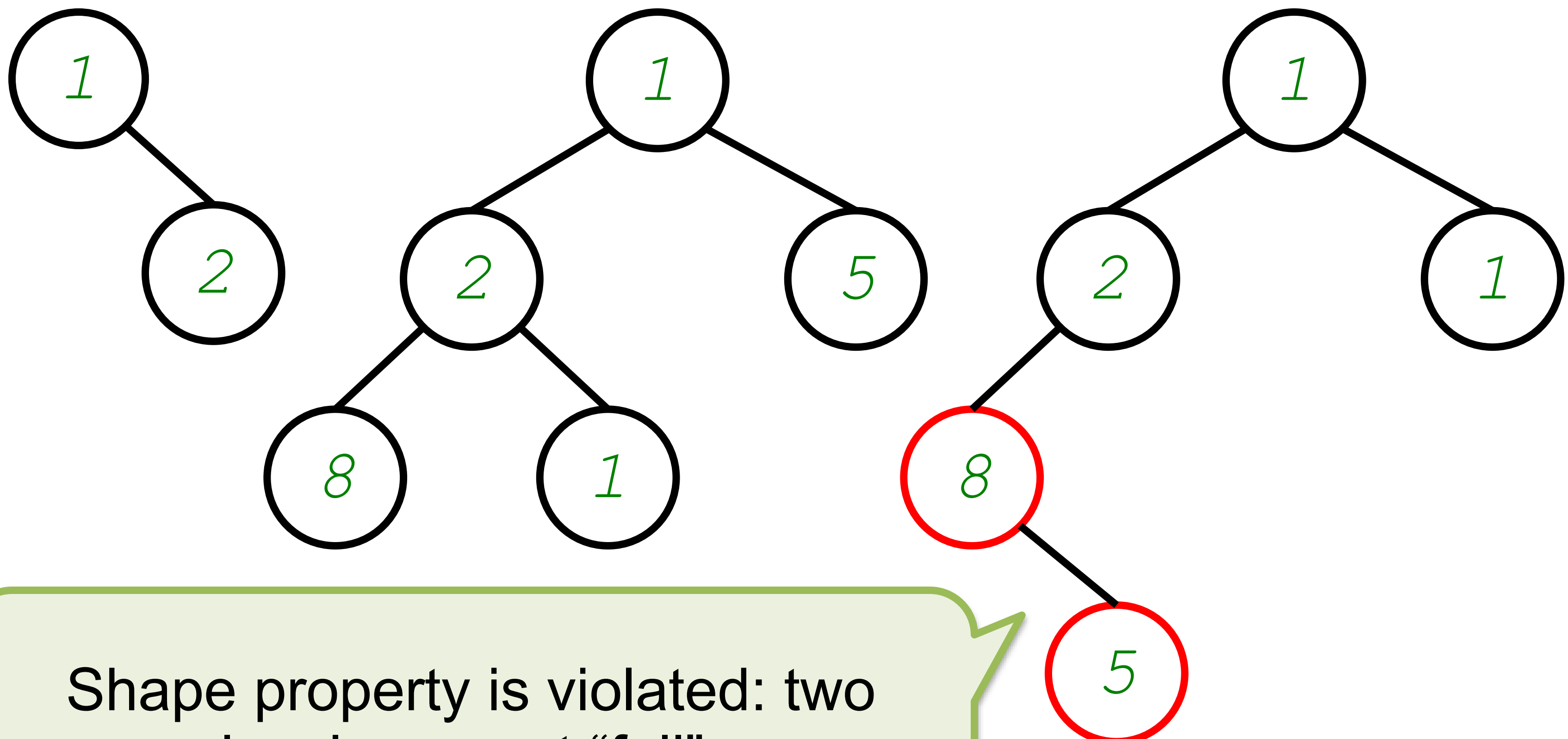
Shape property is violated here: not all nodes at the bottom level are as far left as possible.

Non-Examples of Heaps



Ordering property is violated here: value is out of order with that of its right child.

Non-Examples of Heaps



Shape property is violated: two levels are not “full”.

Heapsort

- A heap can be used to represent the values in a `SortingMachine`, as follows:
  - In `changeToExtractionMode`, arrange all the values into a heap
  - In `removeFirst`, remove the root, and adjust the slightly mutilated heap to make it a heap again

Heapsort

Why should this work?

- A heap can be used t… values in a `SortingMach…`, as follows:
  - In `changeToExtractionMo…`, arrange all the values into a heap
  - In `removeFirst`, remove the root, and adjust the slightly mutilated heap to make it a heap again

How `removeFirst` Can Work

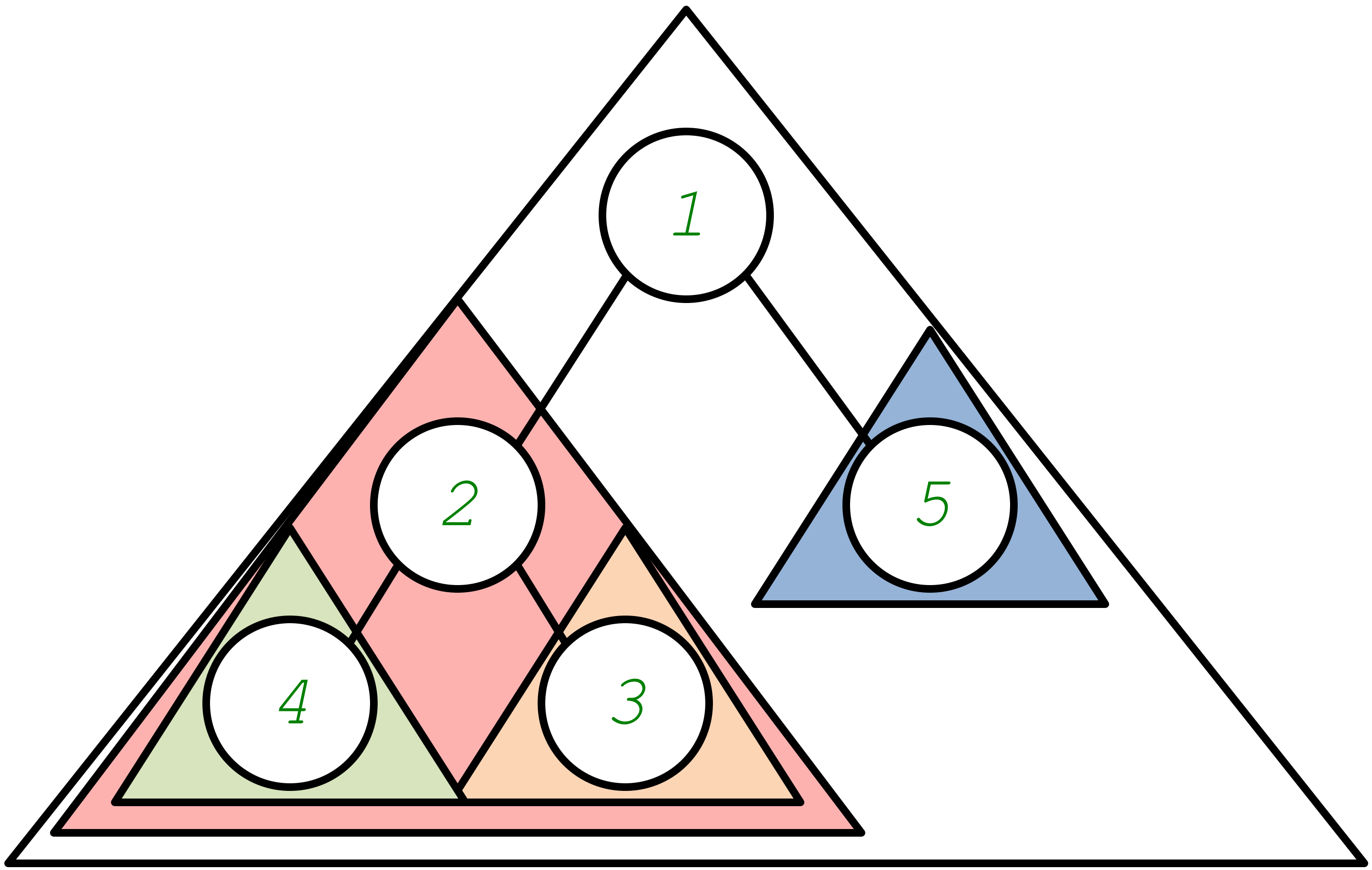
- If the root is the only node in the heap, then after removing it, what remains is already a heap; nothing left to do
- If the root is not the only node, then removing it leaves an “opening” that must be filled by moving some other value in the heap into the opening

How `removeFirst` Can Work

The question is: which one?

- If the root is the only node in ... then after removing it, what remains ... already a heap; nothing left to do
- If the root is not the only node, then removing it leaves an "opening" that must be filled by moving some other value in the heap into the opening

Example: A First Attempt



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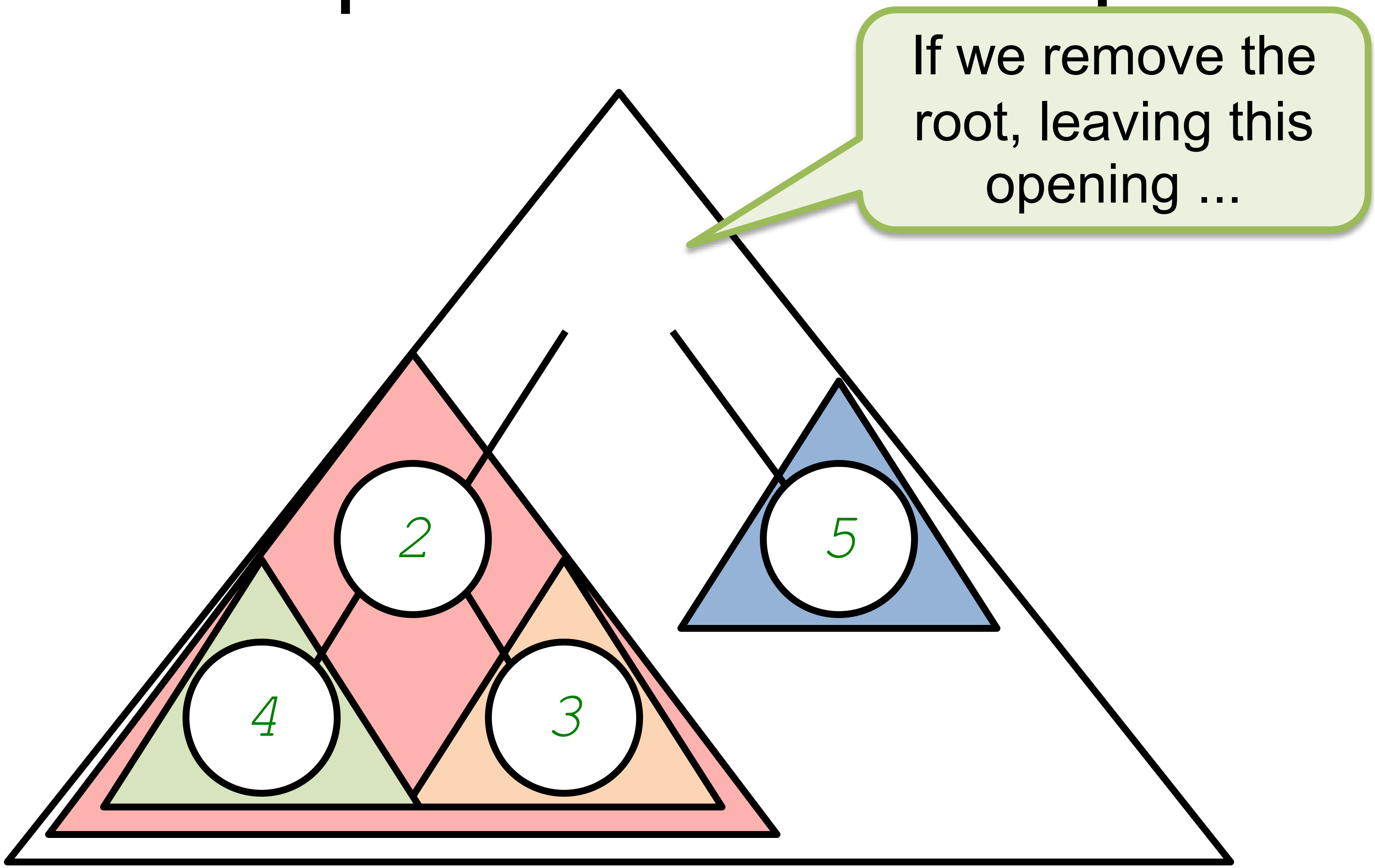

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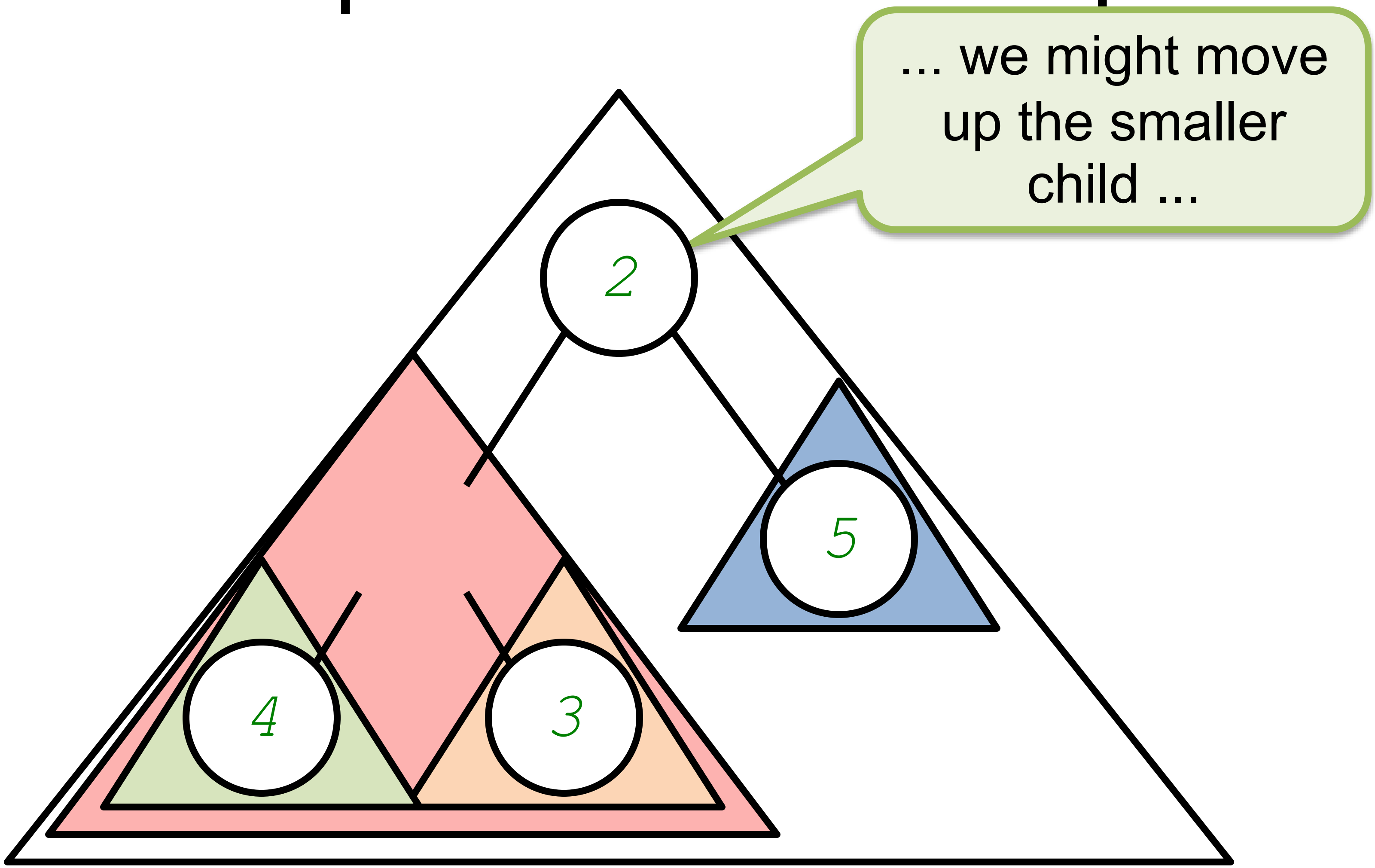

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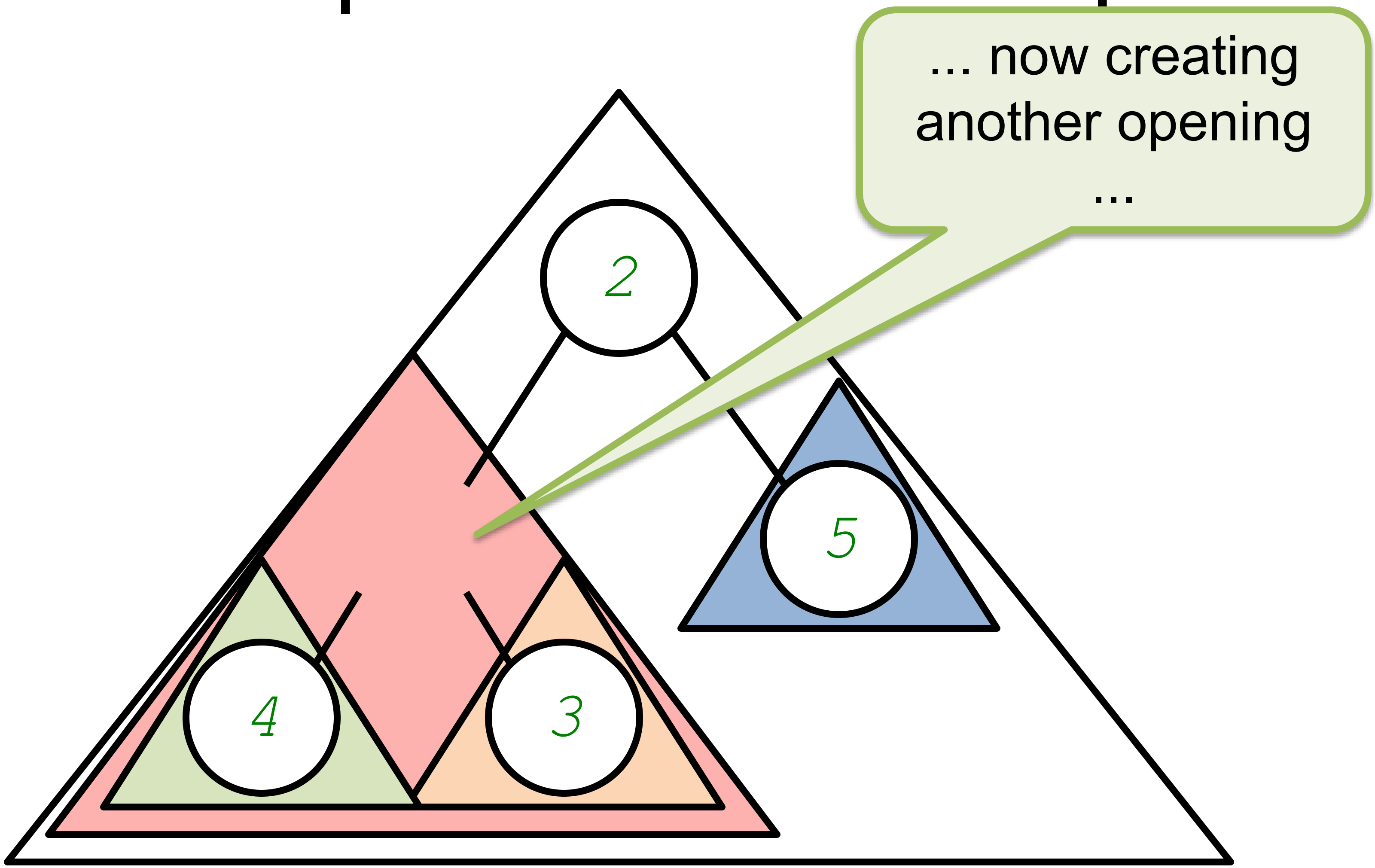

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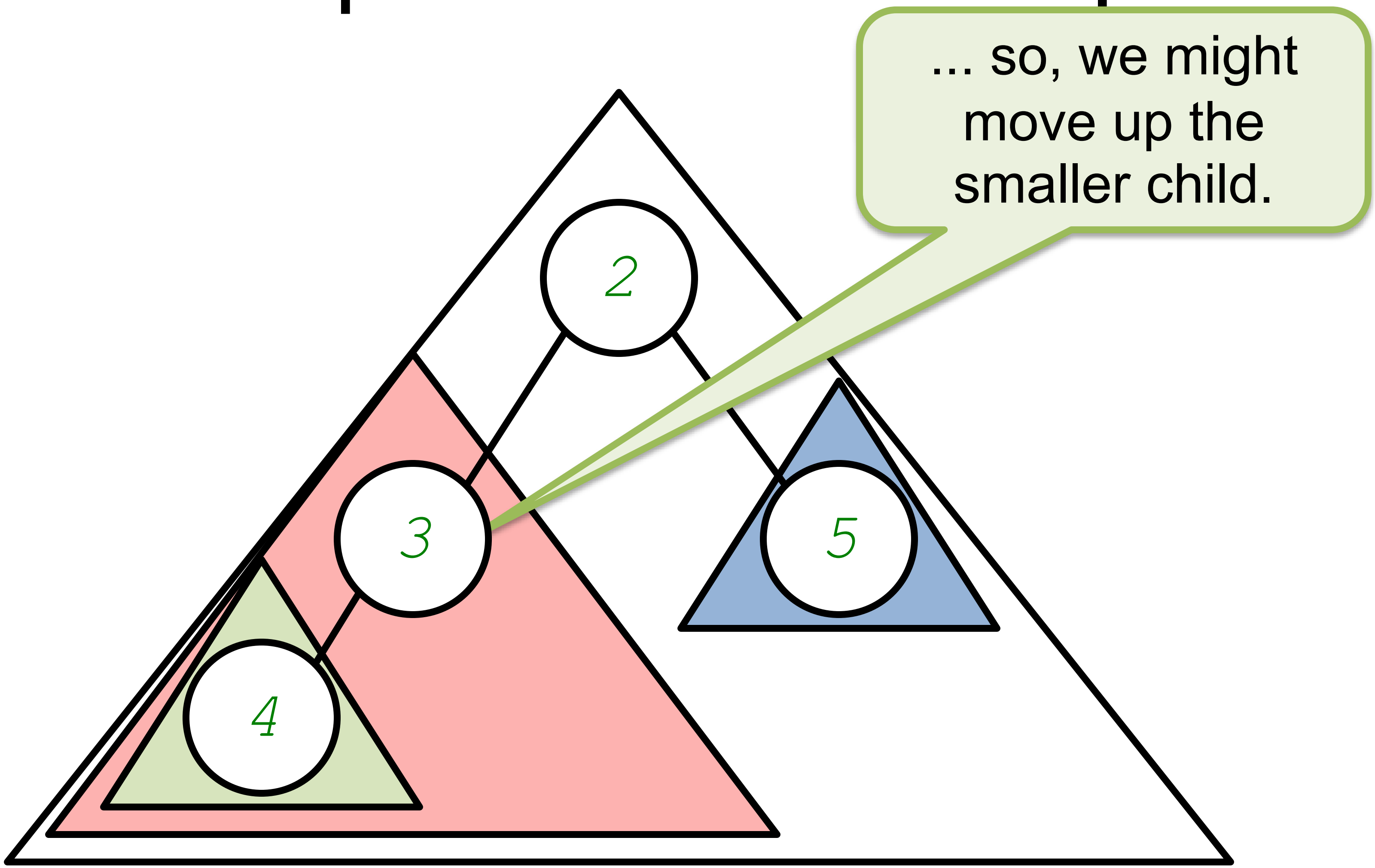

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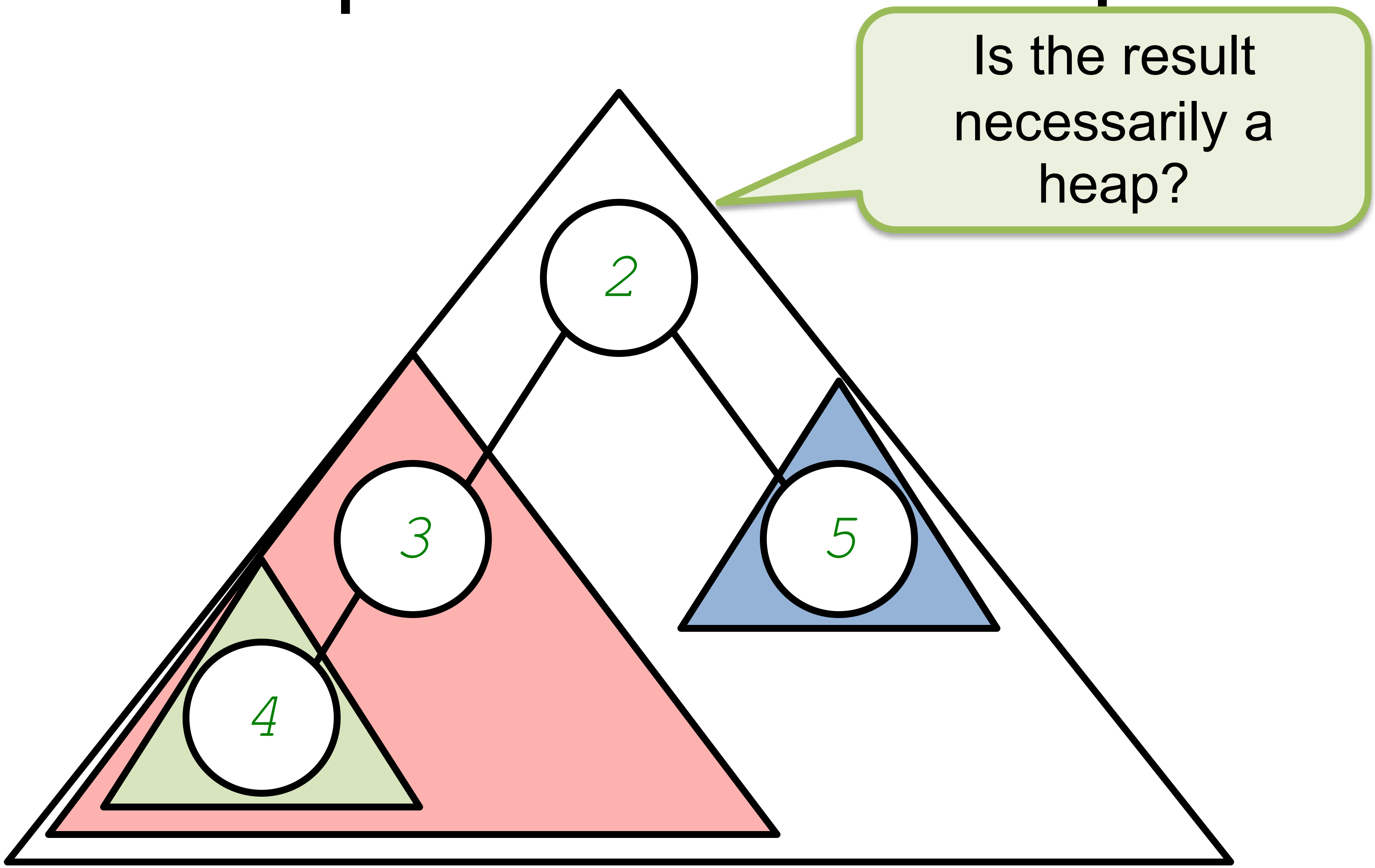
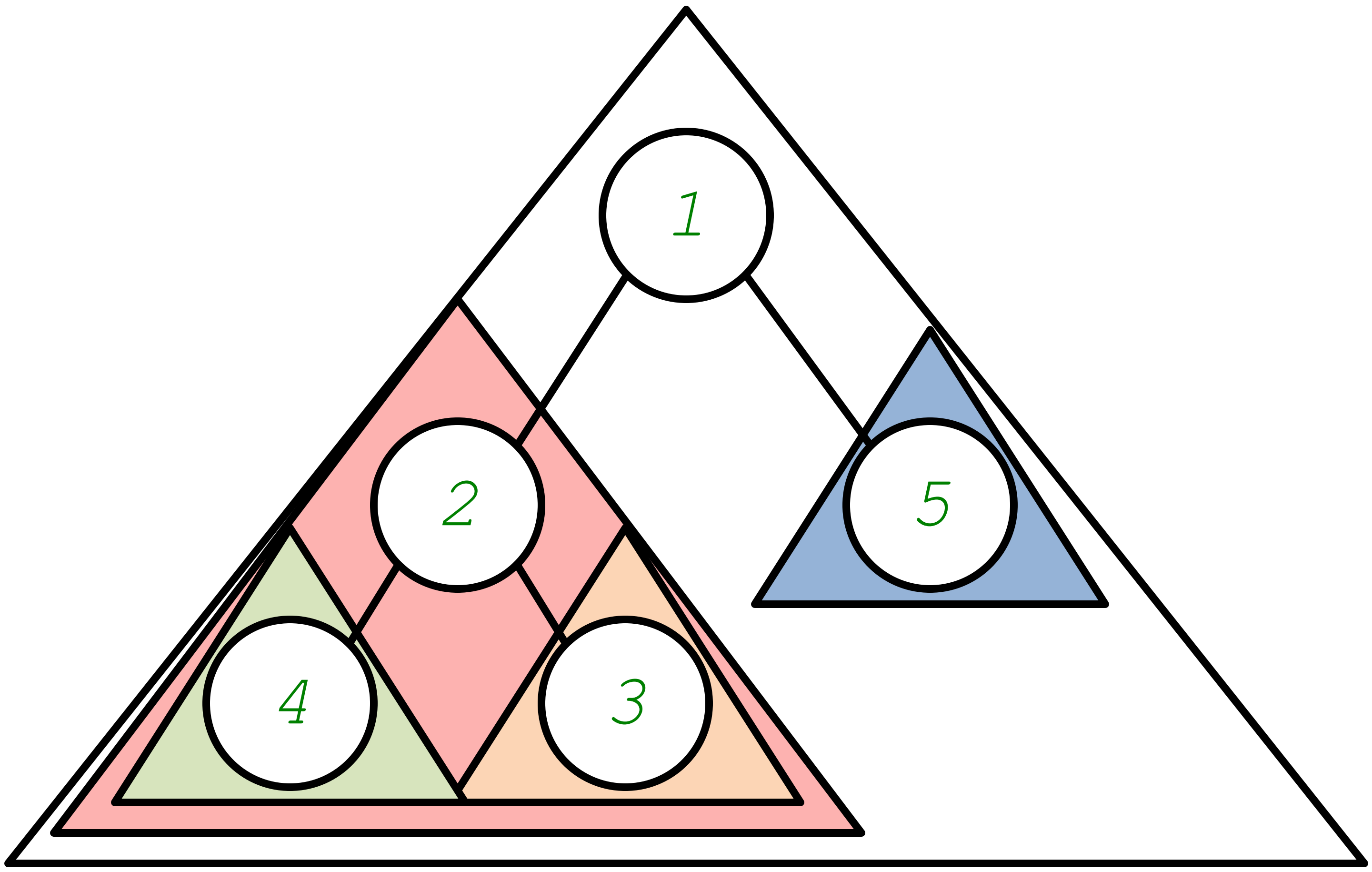


Example: A Second Attempt



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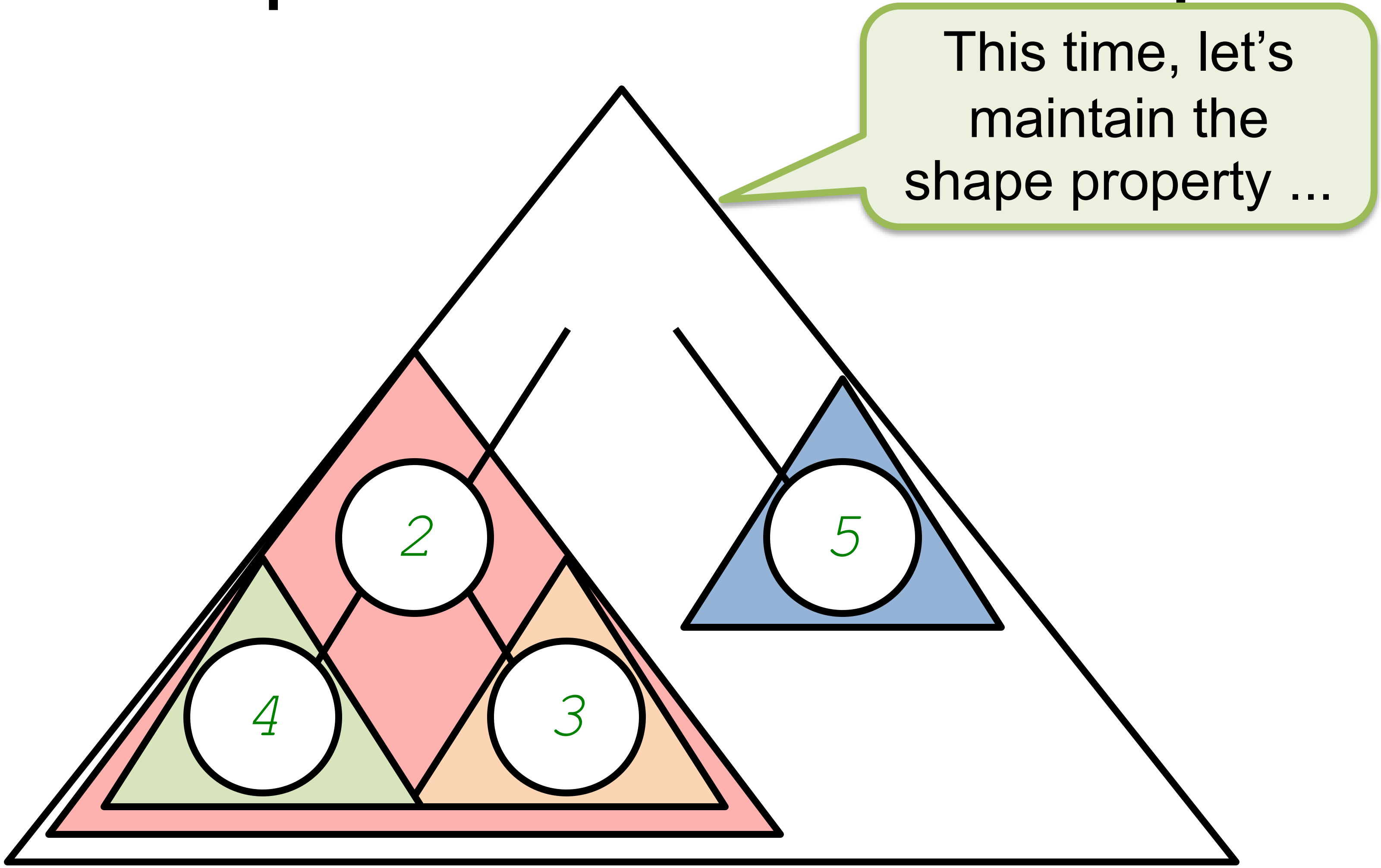

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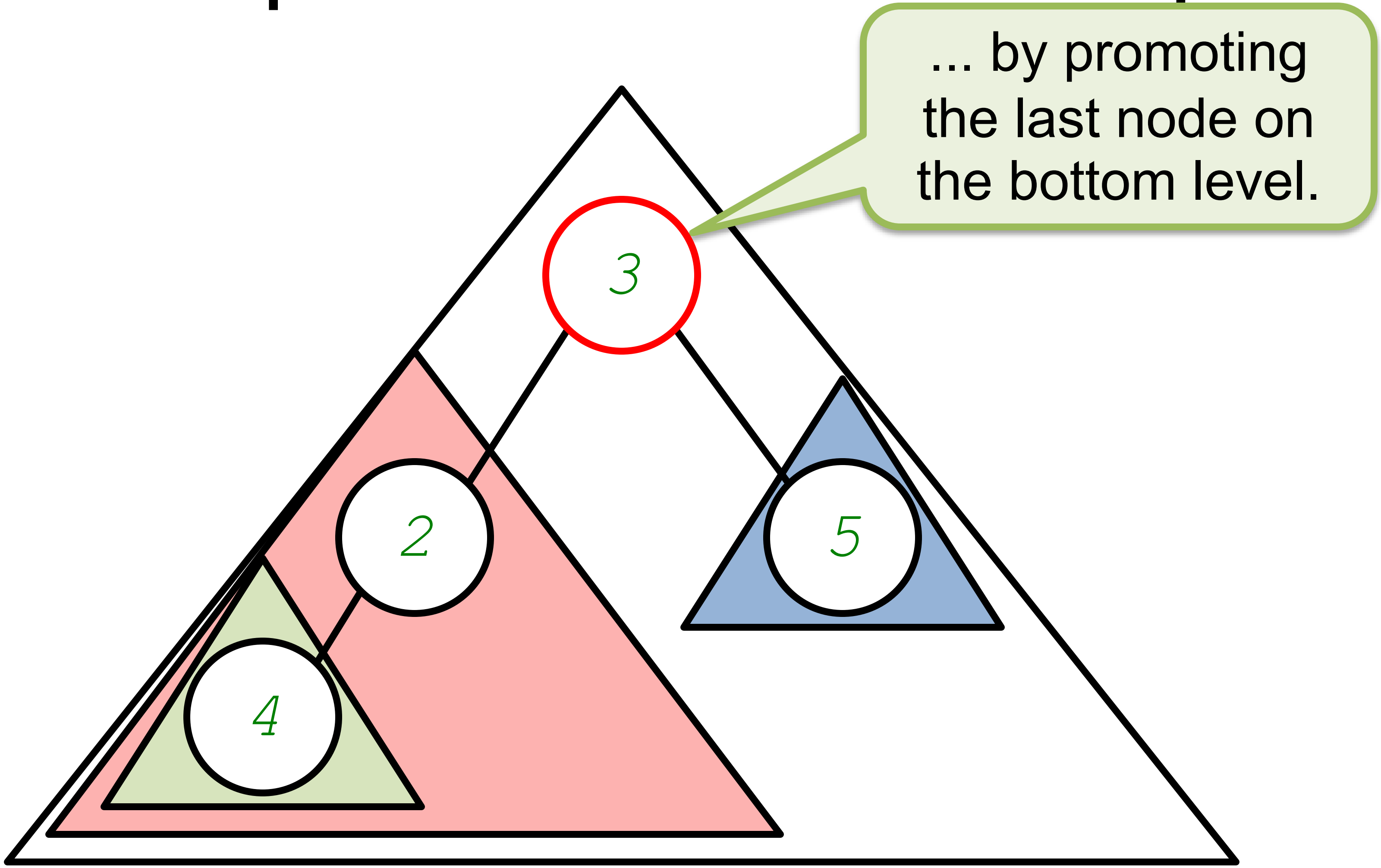

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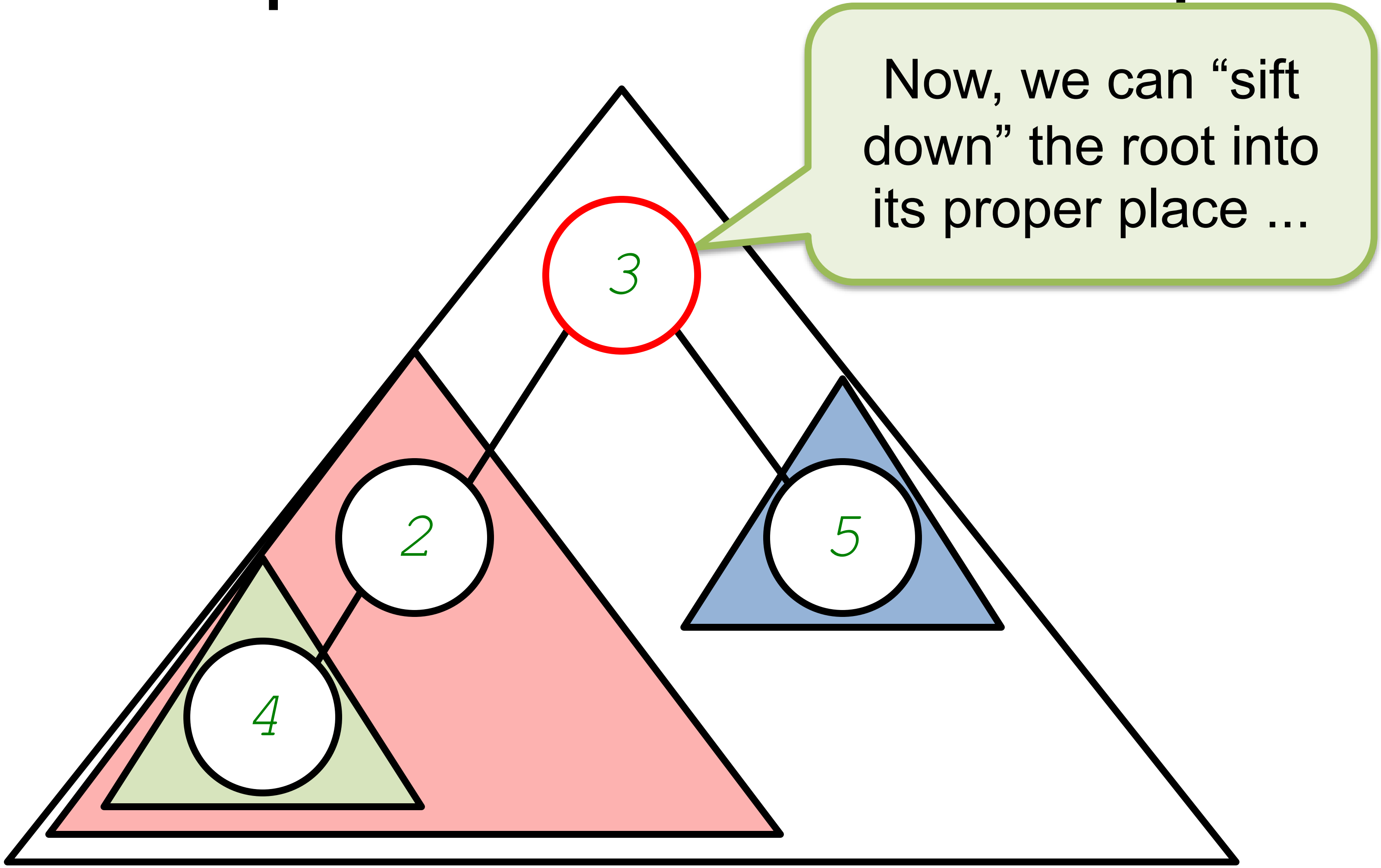

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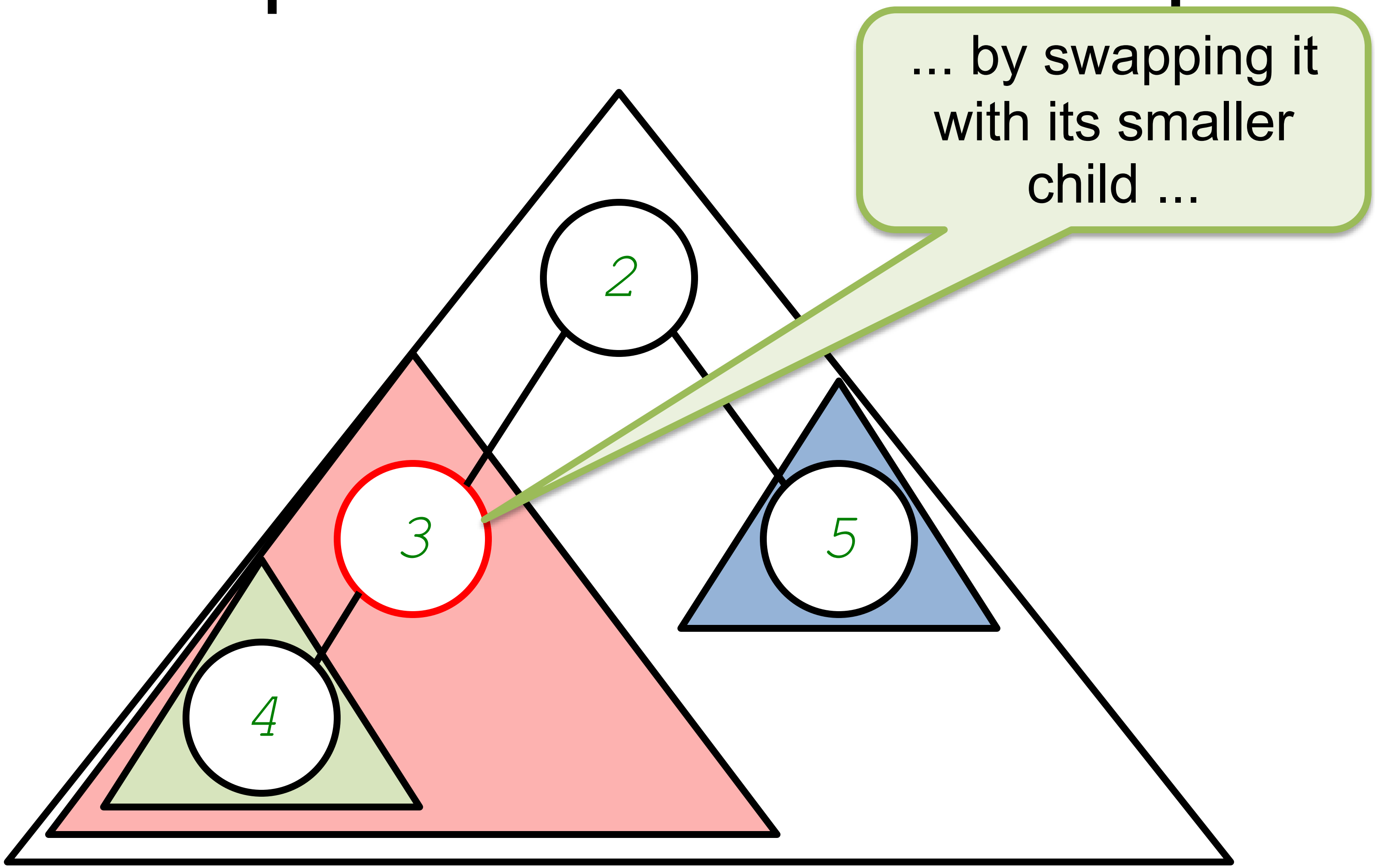

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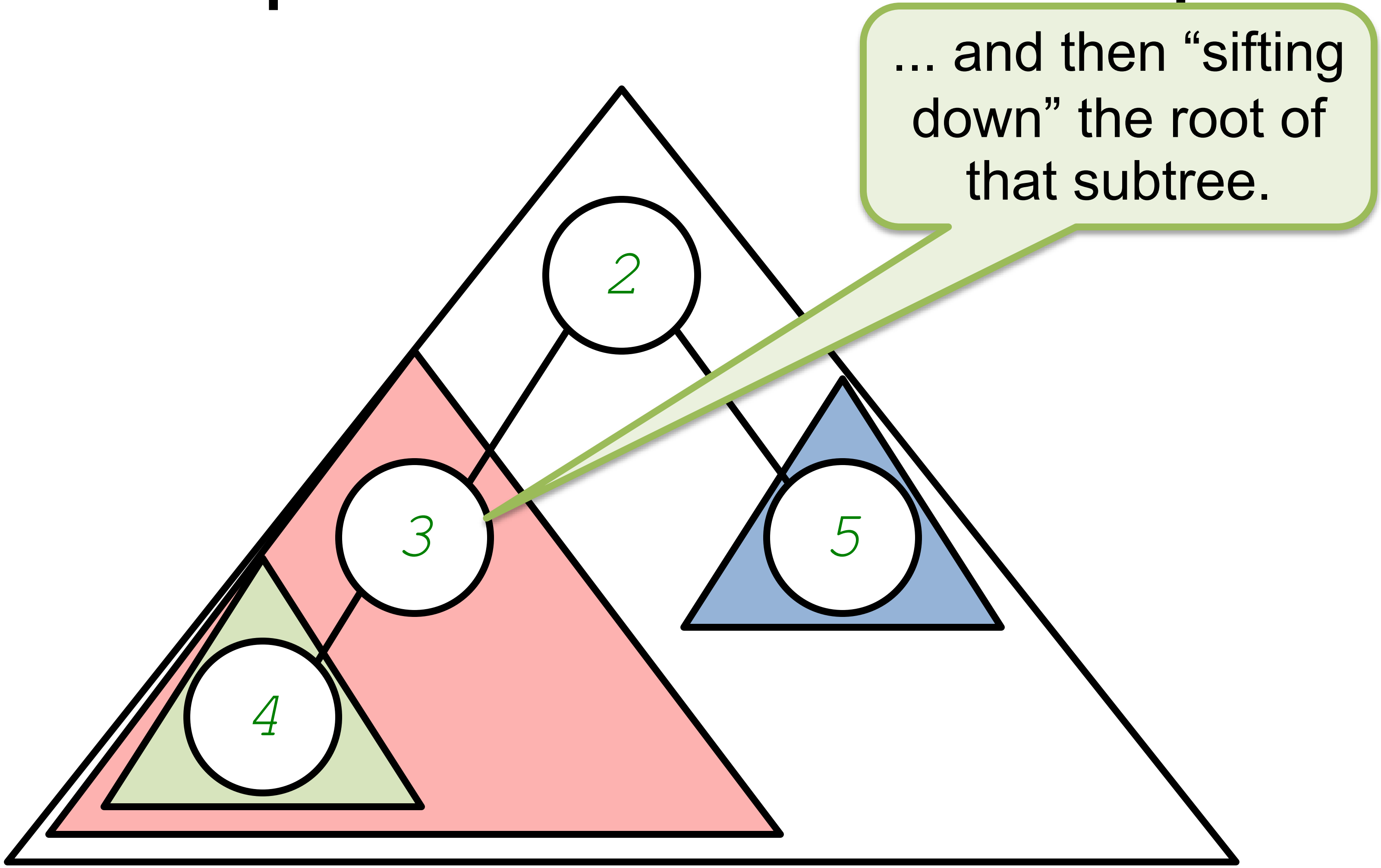

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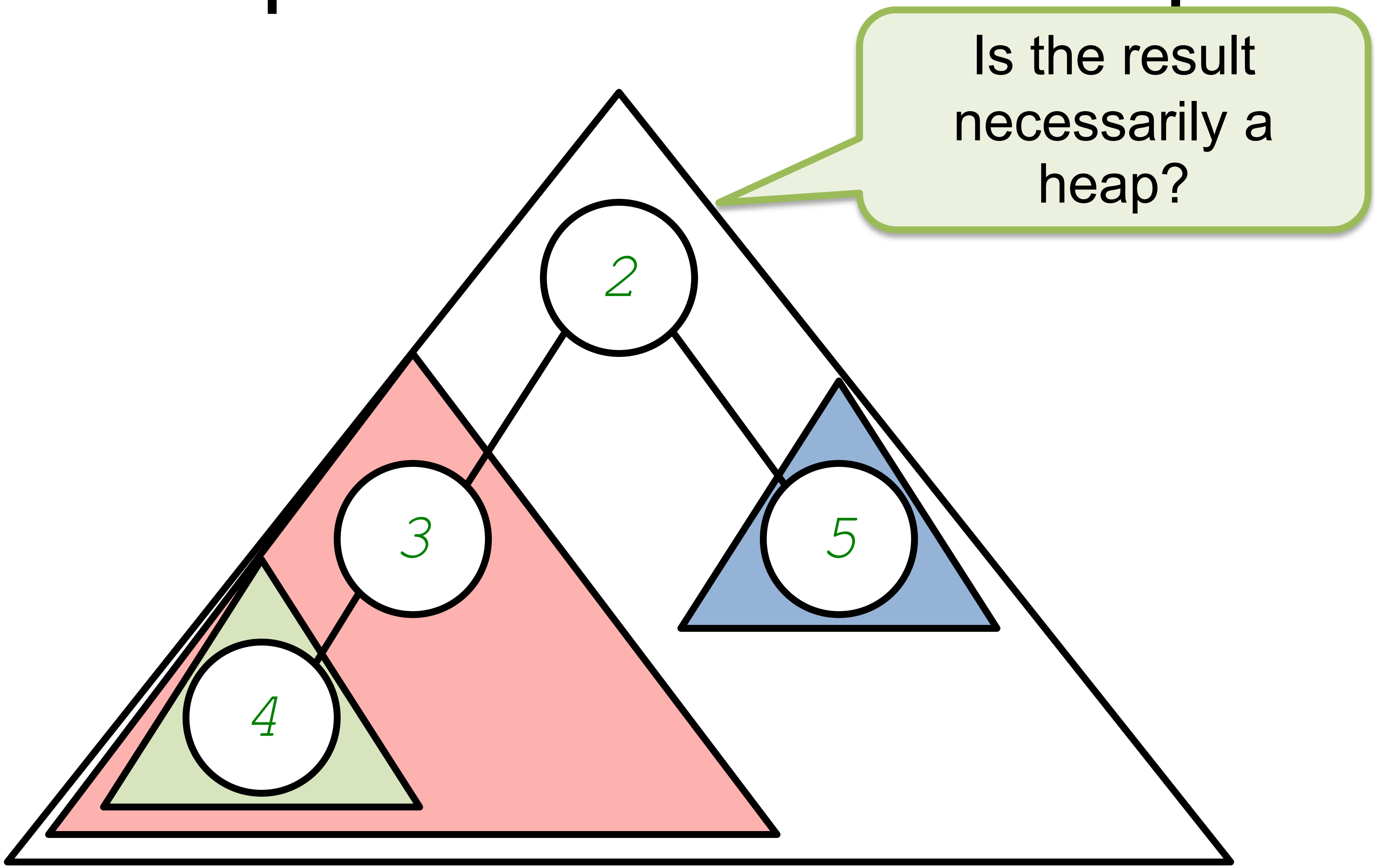


Pseudo-Contract

```
/**
 * Restores a complete binary tree to be a heap
 * if only the root might be out of place.
 * @updates t
 * @requires
 * [t is a complete binary tree]  and
 * [both subtrees of the root of t are heaps]
 * @ensures
 * [t is a heap with the same values as #t]
 */
public static void siftDown (BinaryTree<T> t)
{...}
```

Building a Heap In the First Place

- Suppose we have n values in a complete binary tree, but they are arranged without regard to the heap ordering
- How can we “heapify” it?

Pseudo-Contract

```
/**
 * Makes a complete binary tree into a heap.
 * @updates t
 * @requires
 * [t is a complete binary tree]
 * @ensures
 * [t is a heap with the same values as #t]
 */
public static void heapify (BinaryTree<T> t)
{...}
```

Hint

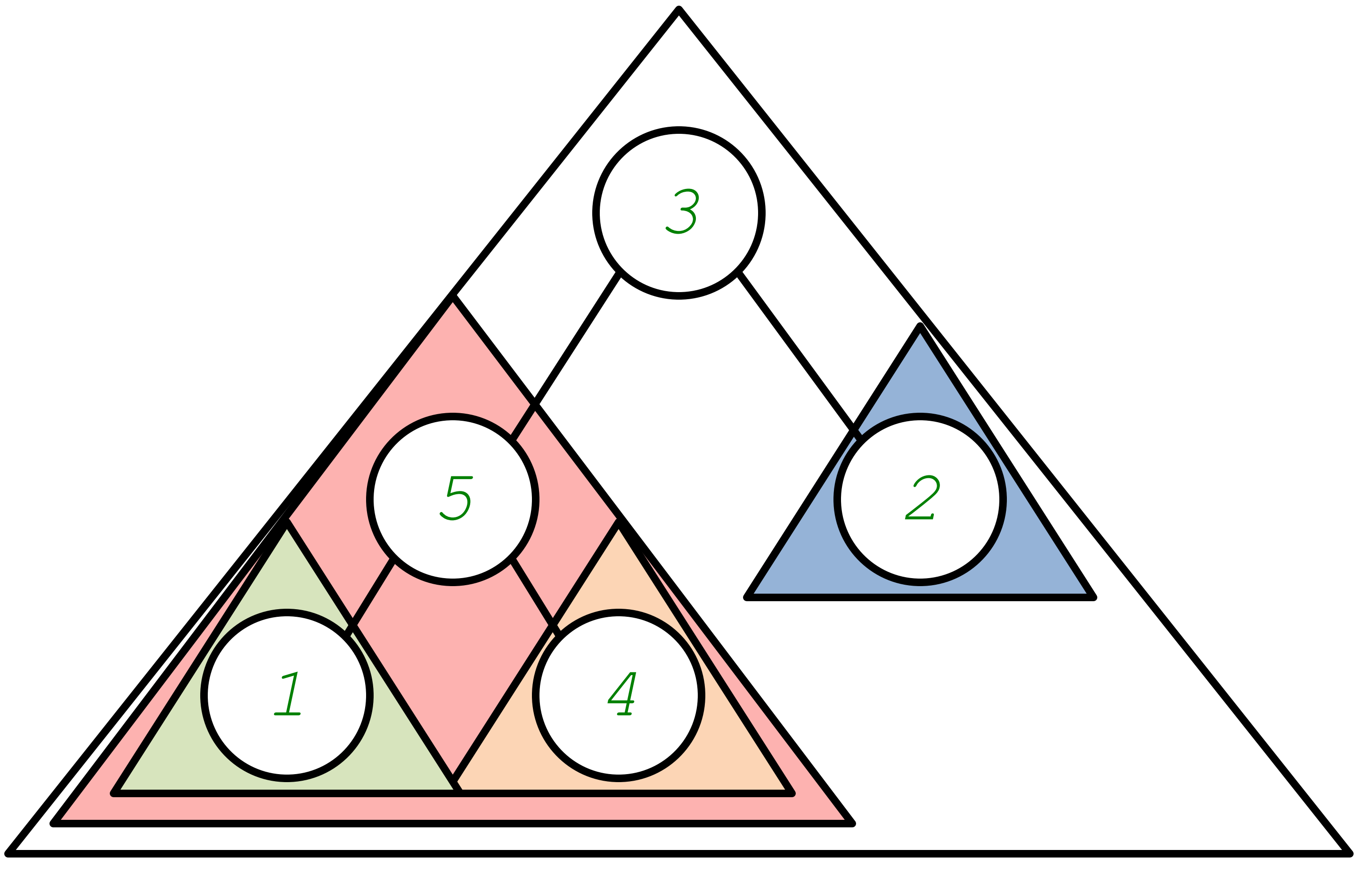
- To see how you might implement `heapify`, compare the contracts of `siftDown` and `heapify`
- The only difference: before we can call `siftDown` to make a heap, both subtrees of the root must already be heaps
  - Once they are heaps, just a call to `siftDown` will finish the job

Hint

How do we make the subtrees into heaps?

- To see how you might imple
  `heapify`, compare the contracts of `siftDown` and `heapify`
- The only difference: before we can call `siftDown` to make a heap, both subtrees of the root must already be heaps
  - Once they are heaps, just a call to `siftDown` will finish the job

Example



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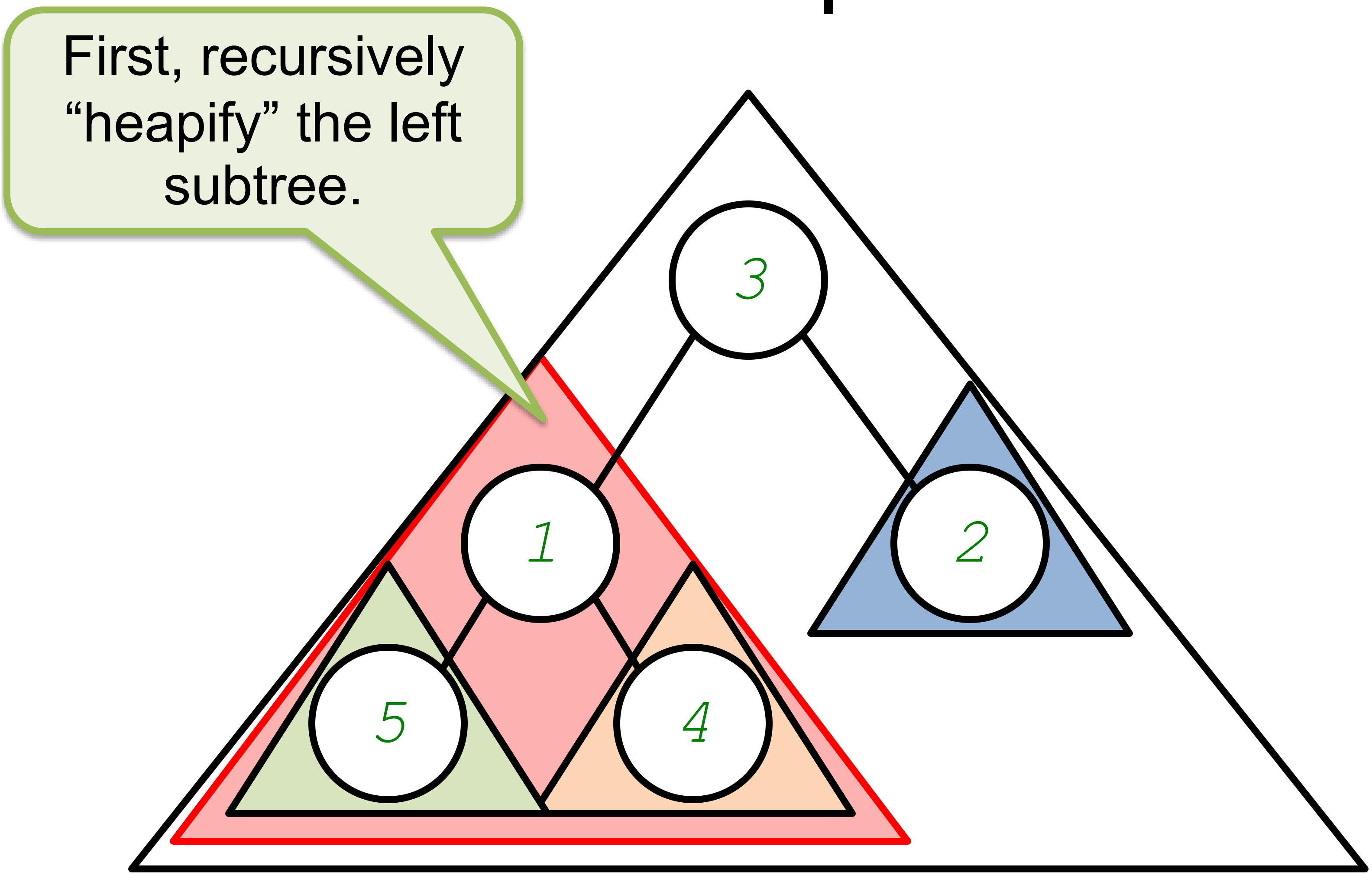

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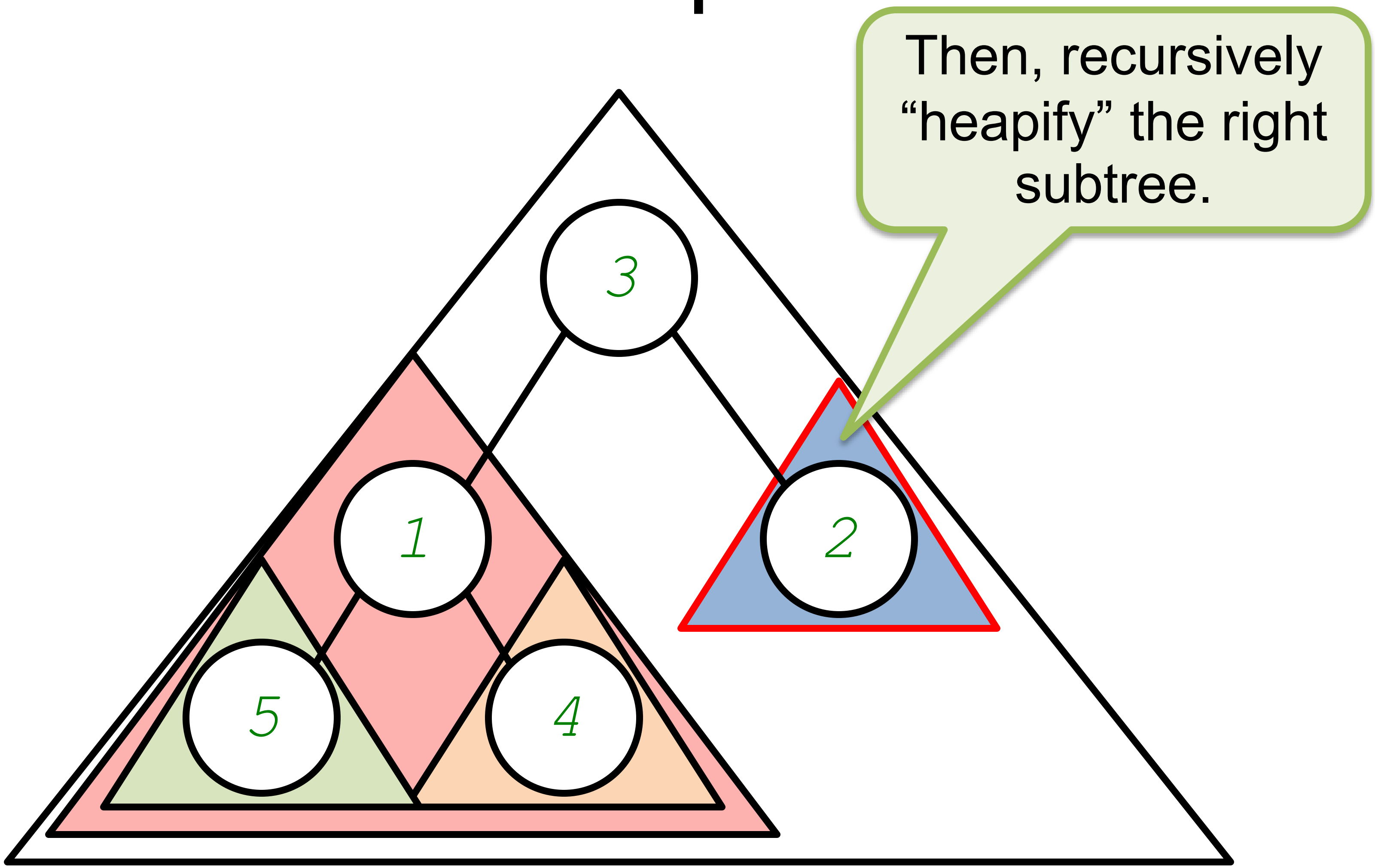
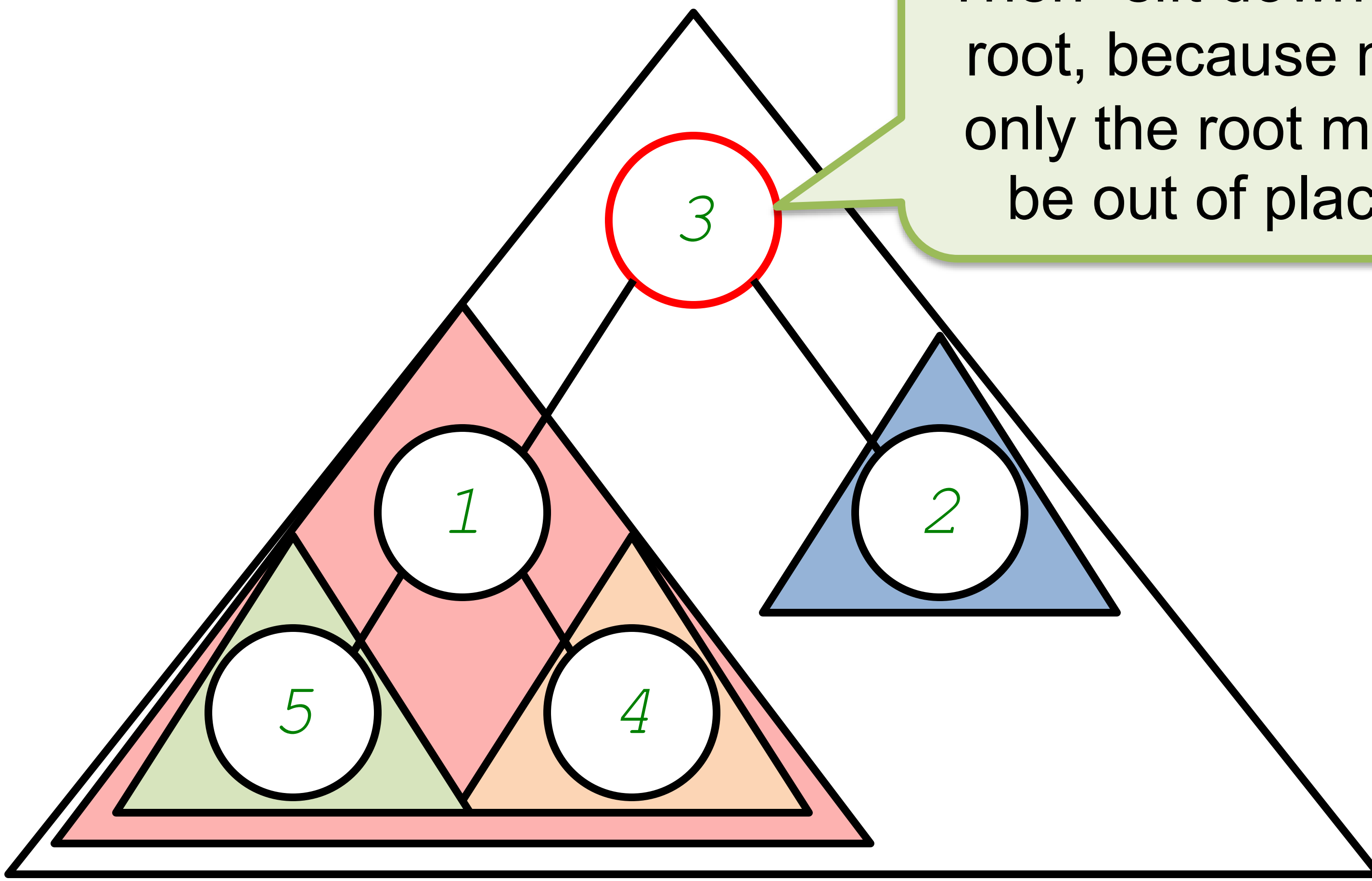


Example

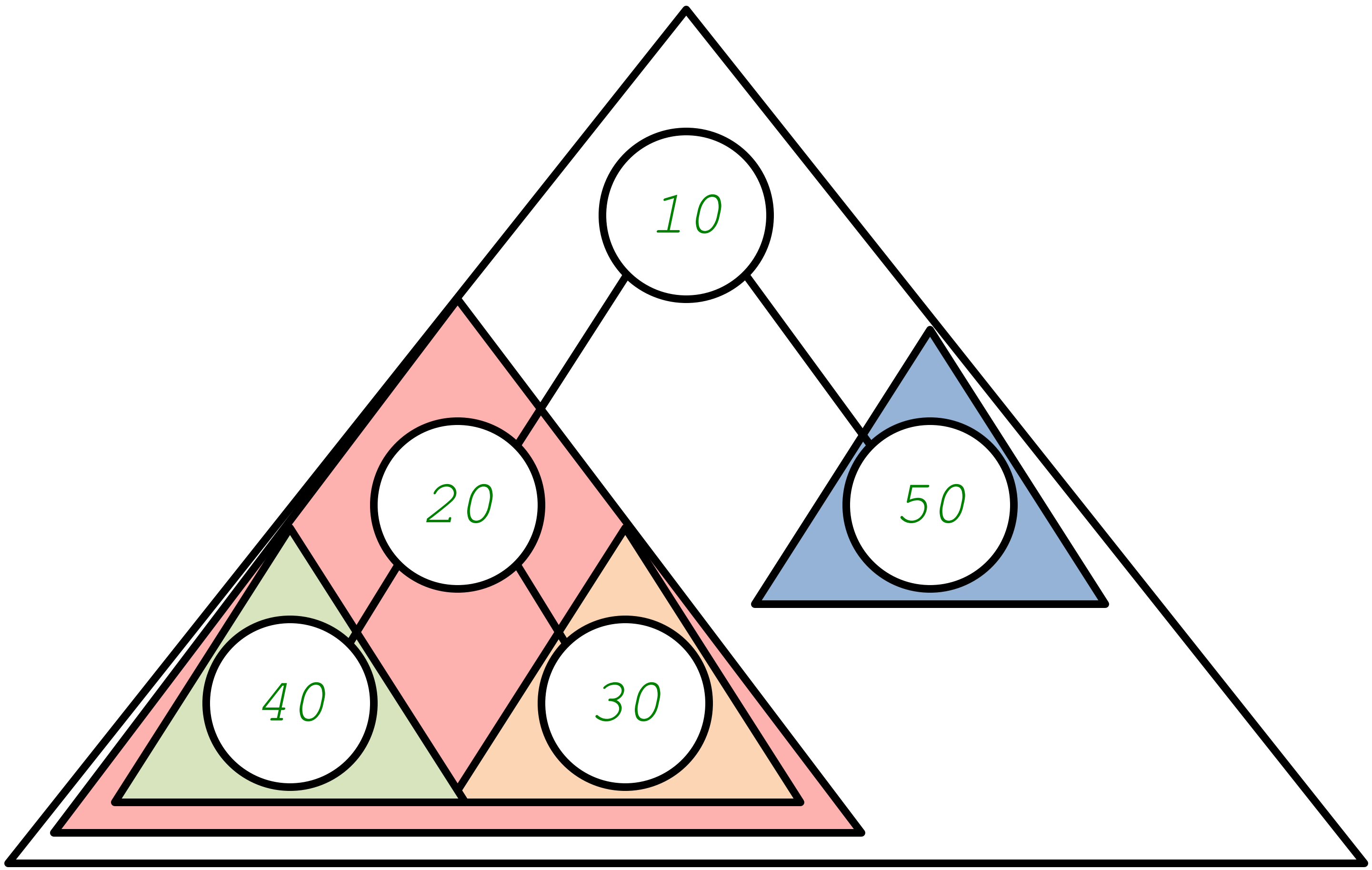


Then "sift down" the root, because now only the root might be out of place.

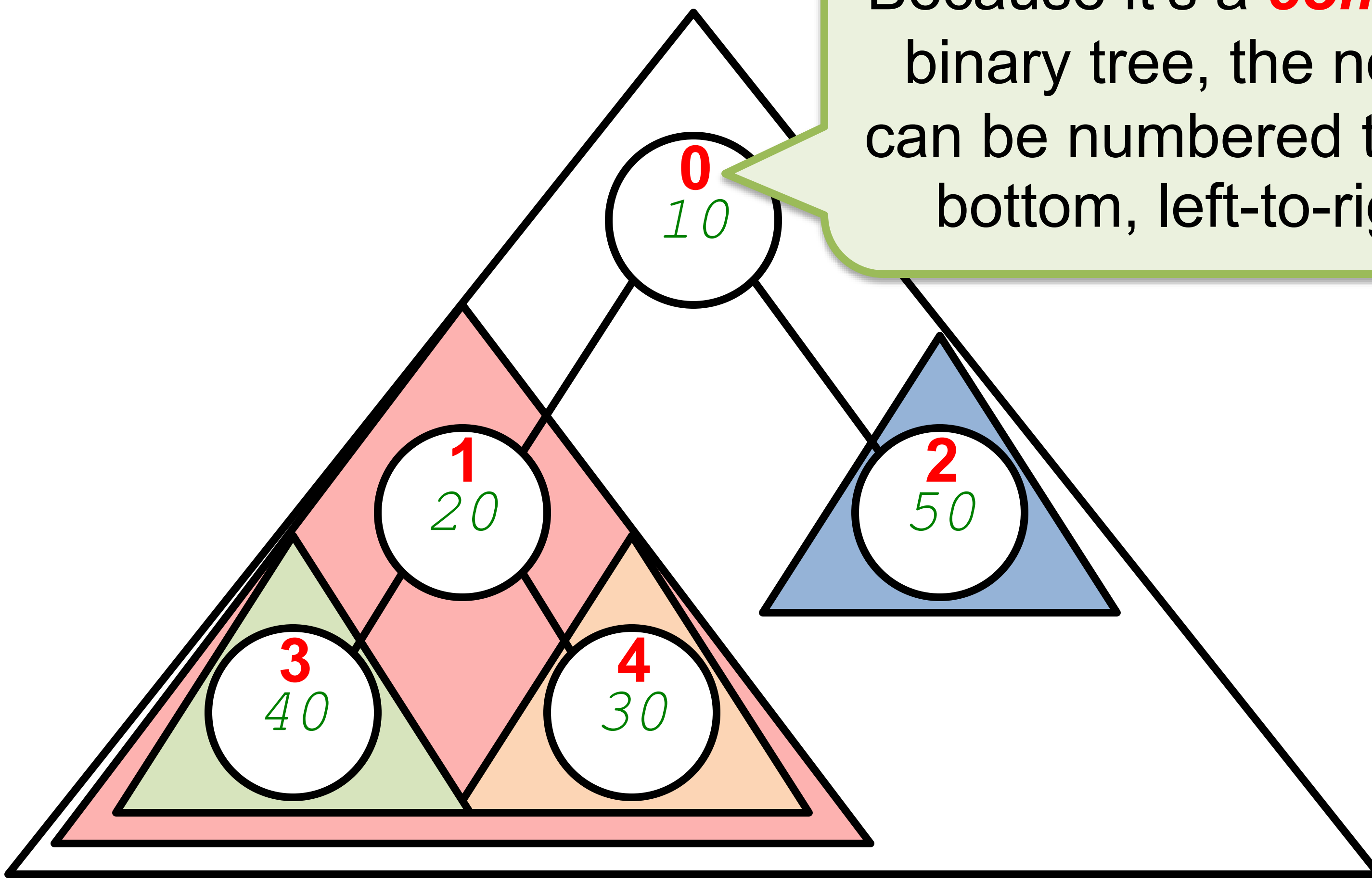
Embedding a Heap in an array

- While one could represent a heap using a `BinaryTree<T>` (as suggested in the pseudo-contracts above), it is generally not done this way
- Instead, a heap is usually represented “compactly” using an array of `T`

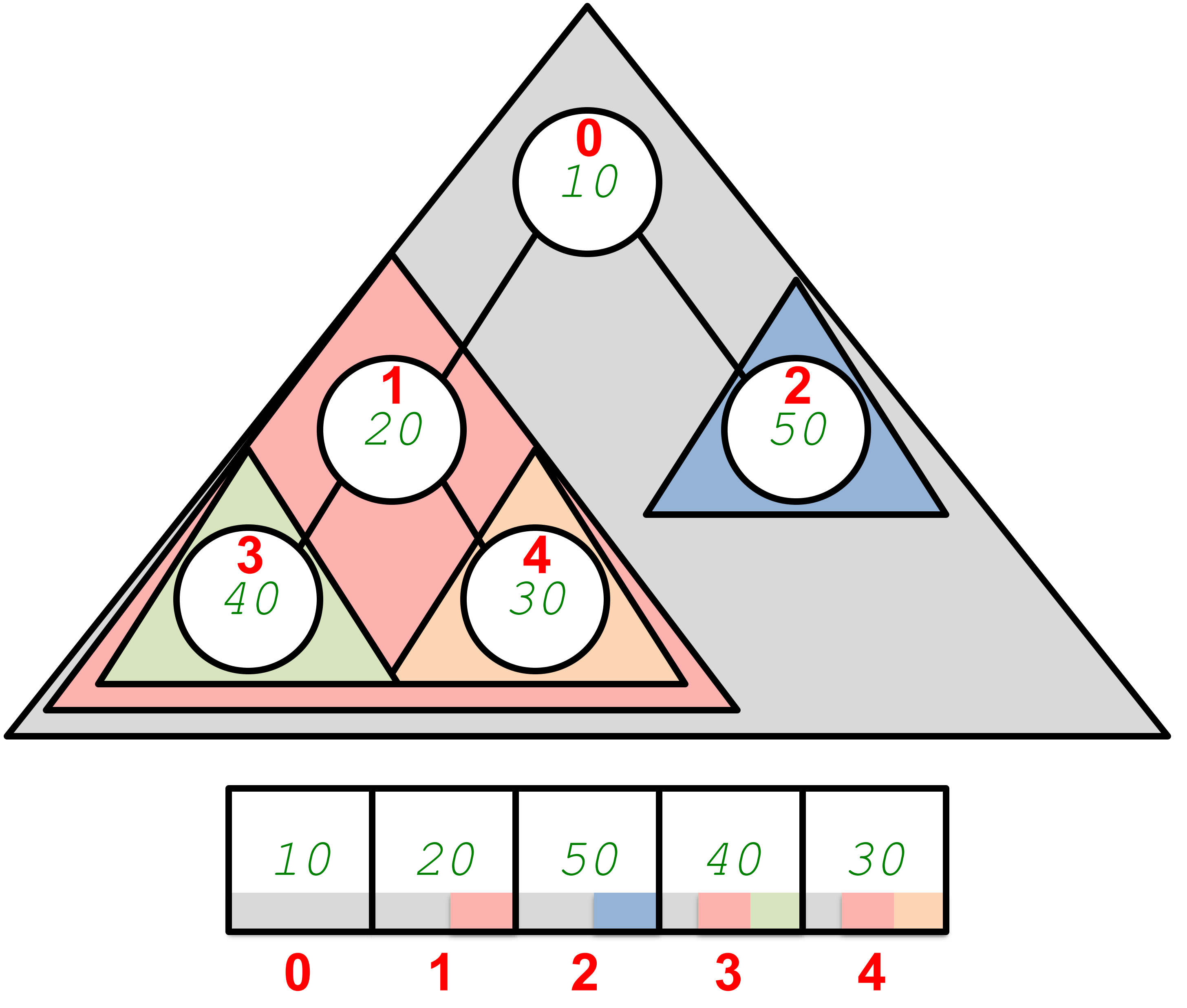
Interpreting an array as a Heap



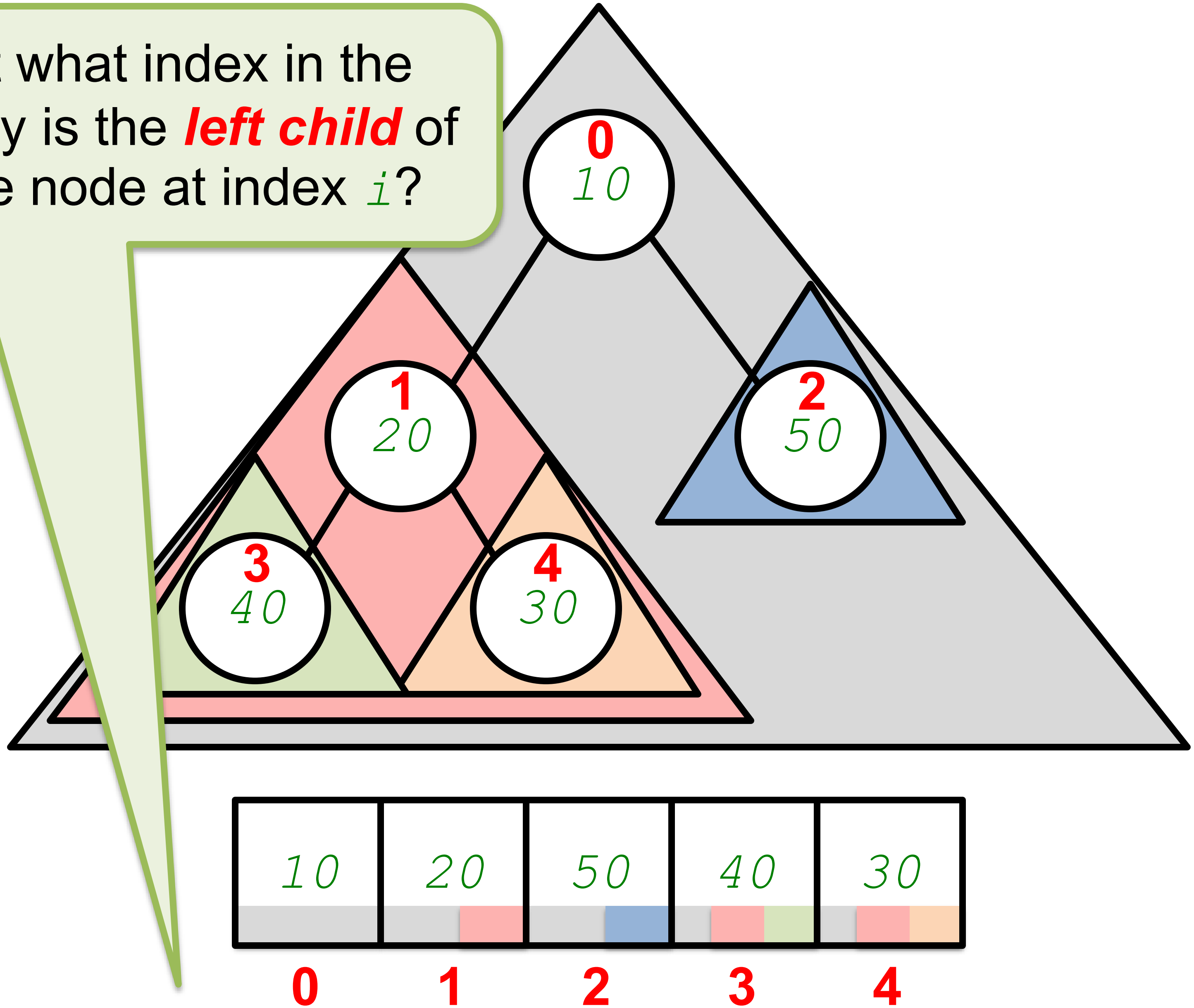
Interpreting an array as a Heap



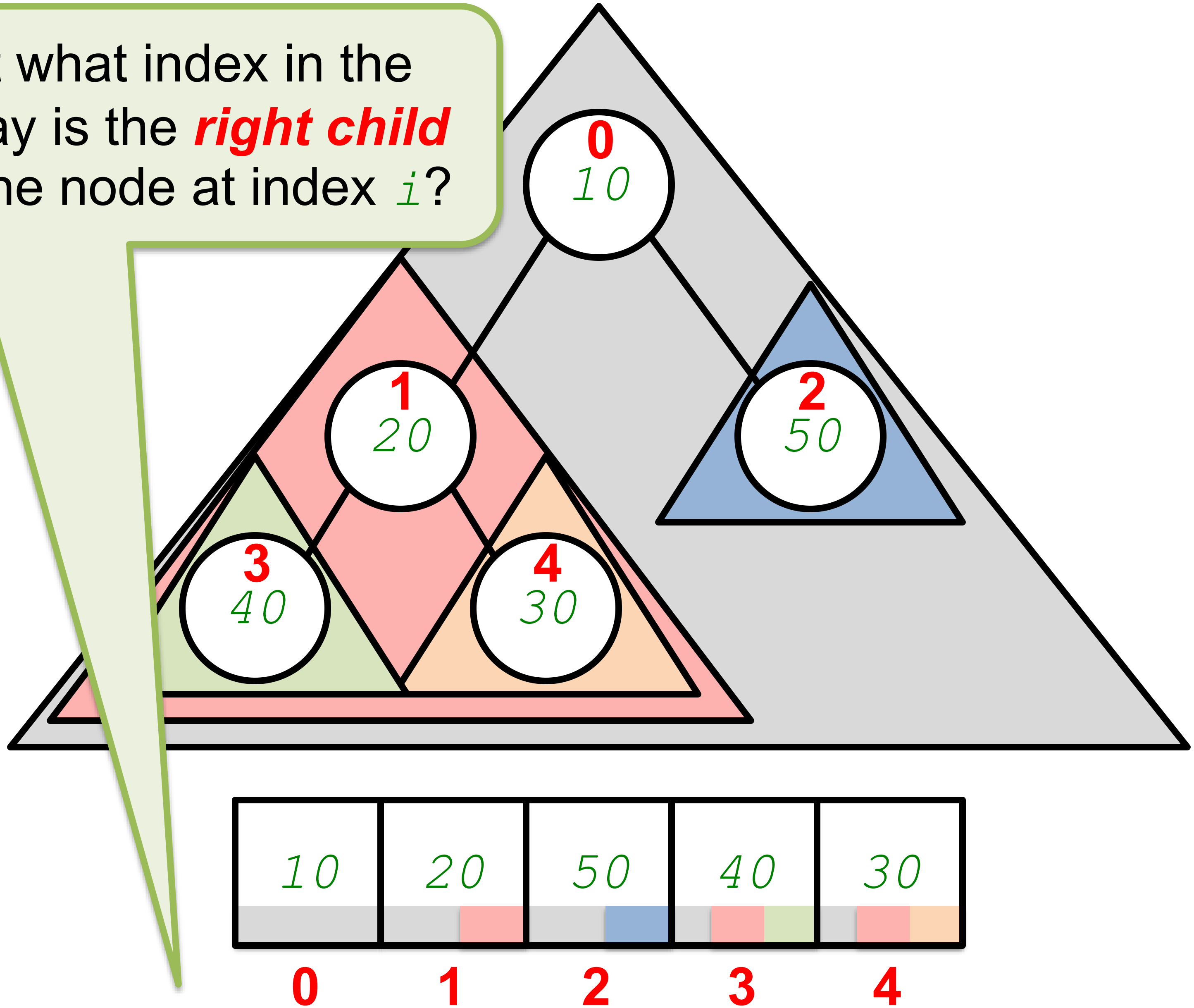
Because it's a ***complete*** binary tree, the nodes can be numbered top-to-bottom, left-to-right.


0
10
1
20
2
50
3
40
4
30
10
20
50
40
30
0
1
2
3
4

At what index in the array is the ***left child*** of the node at index `i`?



At what index in the array is the ***right child*** of the node at index `i`?



Resources

- Wikipedia: Heapsort
  - http://en.wikipedia.org/wiki/Heapsort
- Wikipedia: Heap (data structure)
  - http://en.wikipedia.org/wiki/Heap_(data_structure)
- *Big Java (4th ed)*, Sections 16.8, 16.9
  - https://library.ohio-state.edu/record=b8540788~S7