Binary Search Trees
Faster Searching

• The `BinaryTree` component family can be used to arrange the labels on binary tree nodes in a variety of useful ways.

• A common arrangement of labels, which supports searching that is much faster than linear search, is called a `binary search tree (BST)`.
BSTs Are Very General

• BSTs may be used to search for items of any type $T$ for which one has defined a \textit{total preorder}, i.e., a \textit{binary relation} on $T$ that is \textit{total, reflexive,} and \textit{transitive}
A binary relation on $T$ may be viewed as a set of ordered pairs of $T$, or as a boolean-valued function $R$ of two parameters of type $T$ that is true iff that pair is in the set.

**Total preorder**, i.e., a binary relation on $T$ that is total, reflexive, and transitive.
BSTs Are Very General

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The binary relation $R$ is total whenever:

$$\text{for all } x, y : T \quad (R(x, y) \text{ or } R(y, x))$$
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The binary relation \( R \) is reflexive whenever:

\[
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\]
BSTs Are Very General

• BSTs may be used to search for items of any type $T$ for which one has defined a total preorder, i.e., a binary relation on $T$ that is total, reflexive, and transitive.

The binary relation $R$ is transitive whenever:

for all $x, y, z: T$

(if $R(x, y)$ and $R(y, z)$

then $R(x, z)$)
Simplifications

• For simplicity in the following illustrations, we use only one kind of example:
  – $T = \text{integer}$
  – The ordering is $\leq$

• For simplicity (and because of how we will use BSTs), we assume that no two nodes in a BST have the same labels
Simplifications

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Both these simplifications are inessential: BSTs are not limited to these situations!
BST Arrangement Properties

• A *binary tree* is a BST whenever the arrangement of node labels satisfies these two properties:
  
  1. For every node in the tree, if its label is $x$ and if $y$ is a label in that node’s *left* subtree, then $y < x$
  
  2. For every node in the tree, if its label is $x$ and if $y$ is a label in that node’s *right* subtree, then $y > x$
The Big Picture
Every label $y$ in this tree satisfies $y < x$
The Big Picture

Every label $y$ in this tree satisfies $y > x$
And It’s So Everywhere
Every label $y$ in this tree satisfies $y < x$
And It’s So Everywhere

Every label \( y \) in this tree satisfies \( y > x \)
Examples of BSTs

- Image of two binary search trees with values 1, 2, 3, 4, 5, 6, 7, 8, 9.
Non-Examples of BSTs
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Property 1 is violated here.
Non-Examples of BSTs

Property 1 is violated here.
Non-Examples of BSTs

Property 2 is violated here.
Searching for $x$

- Suppose you are trying to find whether any node in a BST $t$ has the label $x$

- There are only two cases to consider:
  - $t$ is empty
  - $t$ is non-empty
Searching for $x$

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Easy: Report $x$ is not in $t$. 
Searching for $X$
Searching for $x$

Does $x = r$?  
If so, report that $x$ is in $t$.  
If not ...
Searching for $X$

Is $x < r$? If so, report the result of searching for $x$ in this tree. If not ...
Then it must be the case that $x > r$. Report the result of searching for $x$ in this tree.
Why It’s Faster Than Linear Search

• You need to compare to the root of the tree, and then (only if the root is not what you’re searching for) search \textit{either} the left or the right subtree—but not both
  – Compare to linear search, where you might have to look at all the items, which would be equivalent to searching \textit{both} subtrees
Example: Searching for 5
Example: Searching for 5

Does $5 = 8$?
No ...
Is $5 < 8$? Yes, so report the result of searching for $5$ in this tree.
Recursion

• Searching the left subtree at this point simply involves making a recursive call to the method that searches a BST

• Against our usual advice about recursion, let’s trace into that call and see what happens
  – Why? Because some people, e.g., interviewers, may expect you to understand BSTs without mentioning recursion/induction
Example: Searching for 5
Example: Searching for 5

Does $5 = 3$?
No ...

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Is $5 < 3$?
No ...
Then it must be the case that $5 > 3$. Report the result of searching for 5 in this tree.
It’s Another Recursive Call

• Let’s continue tracing into calls ...
Example: Searching for 5
Does $5 = 6$?
No ...
Is $5 < 6$?
Yes, so report the result of searching for $5$ in the (empty) left subtree.
The Recursion Stops Here

• Remember, we already noted that when searching for something in an empty tree, we can simply report it is not there

• No new recursive call results
How many nodes did the algorithm visit, and compare labels to 5? At worst, how many could it be?
Example: Searching for 5

What about in this tree?
Wait! How Can This Work?

• With the BinaryTree components, there are no methods to “move down the tree”
• This is why recursion is crucial
  – To search a subtree, you disassemble the original tree, search in one of the subtrees, and then (re)assemble it before returning the answer
Refined Searching for $x$
Searching for $x$

Does $x = r$?
If so, report that $x$ is in $t$.
If not ...
Searching for $x$

Disassemble $t$ into the root and its two subtrees.
Searching for $x$

Is $x < r$?
If so, remember the result of searching for $x$ in this tree.
If not ...
Then it must be the case that $x > r$. Remember the result of searching for $x$ in this tree.
Searching for $X$

Before returning the result of the search, (re)assemble $\tau$ from its parts.
Inserting $x$

• Suppose now you are trying to insert into a BST $t$ the label $x$ (which we assume to be not already in $t$; remember that there are no duplicate labels in $t$)

• There are only two cases to consider:
  – $t$ is empty
  – $t$ is non-empty
Inserting $x$

- Suppose now you are trying to insert into a BST $t$ the label $x$ (which we assume to be not already in $t$; remember that there are no duplicate labels in $t$)
- There are only two cases to consider:
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  - $t$ is non-empty

Easy: Make $x$ the root of the updated $t$. 
Inserting $r$
There is no reason to ask whether $x = r$; why?
Inserting $x$

Is $x < r$? If so, insert $x$ into this tree. If not ...
Then it must be the case that $x > r$. Insert $x$ into this tree.
Removing the Smallest

• Suppose now you are trying to remove from a BST $t$ the smallest label (assuming that $t$ is not empty)

• There is only one case to consider:
  – $t$ is non-empty
Removing the Smallest
Removing the Smallest

Does the root have a non-empty left subtree?
If so, remove the smallest label from this tree.
If not, then $r$ is the smallest label in $t$. Make the right subtree the new value of $t$, and return $r$. 
Removing $x$

• Suppose now you are trying to remove from a BST $t$ the label $x$ (which we assume to be in $t$)

• There is only one case to consider:
  – $t$ is non-empty
Removing $x$
Does $x = r$?
If so, ouch! (Later ...)
If not ...
Is $x < r$?
If so, remove $x$ from this tree.
If not ...
Then it must be the case that $x > r$. Remove $x$ from this tree.
Back to the problematic case:
\[ x = r \], i.e., we need to remove the root of \( t \).
What can we do?
The Idea

• To avoid restructuring the entire tree, we can move to the root some label from further down in the tree that would not violate the BST arrangement properties.

• There are two good possibilities for the label to be moved:
  – The next-smaller label than \( x \)
  – The next-larger label than \( x \)
Leverage Prior Work

• We already know how to remove the smallest label from a tree

• So, it is easier to implement this strategy:
  – Remove the smallest label from the right subtree (because this is the next-larger label after $x$ in the original tree $t$)
  – Make that label the new root of $t$
Leverage Prior Work

• We already know how to remove the smallest label from a tree.

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But first, what if the right subtree is empty?
If the right subtree is empty, then make the left subtree the new value of \( t \).
If the right subtree is not empty, then replace \( x \) in the root with the smallest label from the right subtree.
Resources

- *Big Java, Section 16.5*
  - [http://osu.worldcat.org/title/big-java/oclc/754642794](http://osu.worldcat.org/title/big-java/oclc/754642794)