Loop Invariants: Part 1

Reasoning About Method Calls

- What a method call does is described by its ***contract***
  - Precondition: a property that is true *before* the call is made
  - Postcondition: a property that is true *after* the call returns

Reasoning About Loops

- What a `while` loop does is described by its ***loop invariant***
  – Invariant: a property that is true *every time* the code reaches a certain point—in the case of a loop invariant, the loop condition test

Reasoning About Loops

- What a **`while`** loop does is described by its ***loop invariant***
  - Invariant: a property that is true *every time* the code reaches a certain point—in the ase of a loop invariant, the loop condition te

Why is a loop treated differently than a method call? Simply put, experience shows this is a good way to *think about* loops.

Reasoning About Loops

- What a **`while`** loop does is described by its ***loop invariant***
  - Invariant: a prope that is true *every time* the code reaches a ce point—in the case of a loop invariant, the lo ndition test

Just **`while`** loops?

Yes; the same idea can be applied to **`for`** loops, but some modifications are required.

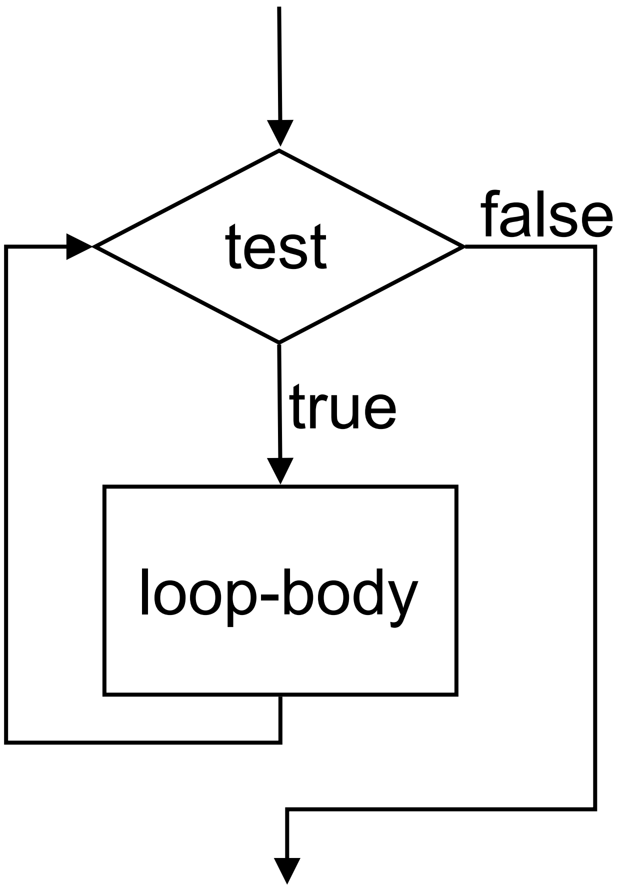
Reasoning About Loops

- What a **`while`** loop does is described by its ***loop invariant***
  - Invariant: a property th is true *every time* the code reaches a certai nt—in the case of a loop invariant, the loo dition test

Since a loop invariant is true every time through the loop, it says what ***does not change***; hence it really says what the loop ***does not do***.

`while` Statement Control Flow

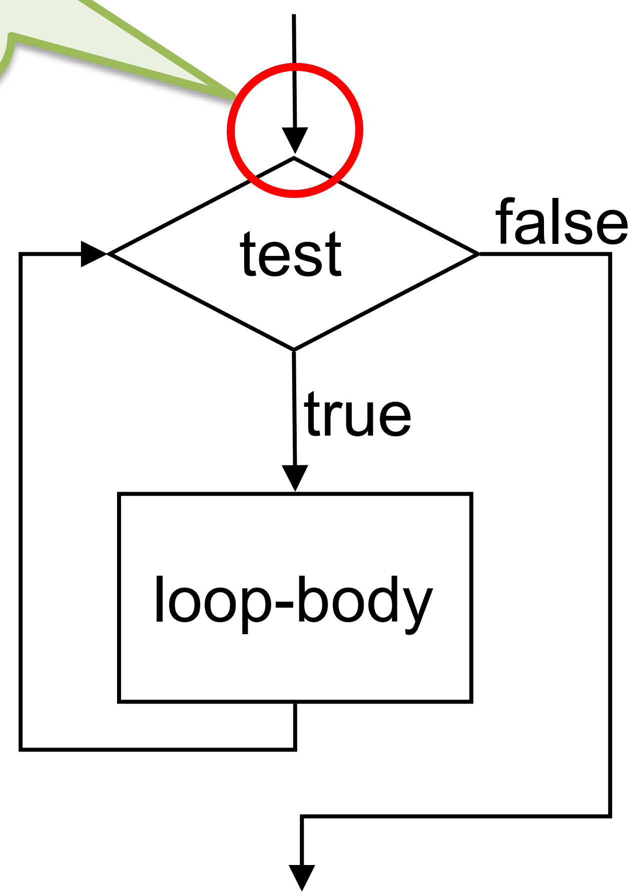
```
while (test) {
  loop-body
}
```



Control Flow

> The loop invariant is a property that is true both here, just before the loop begins...

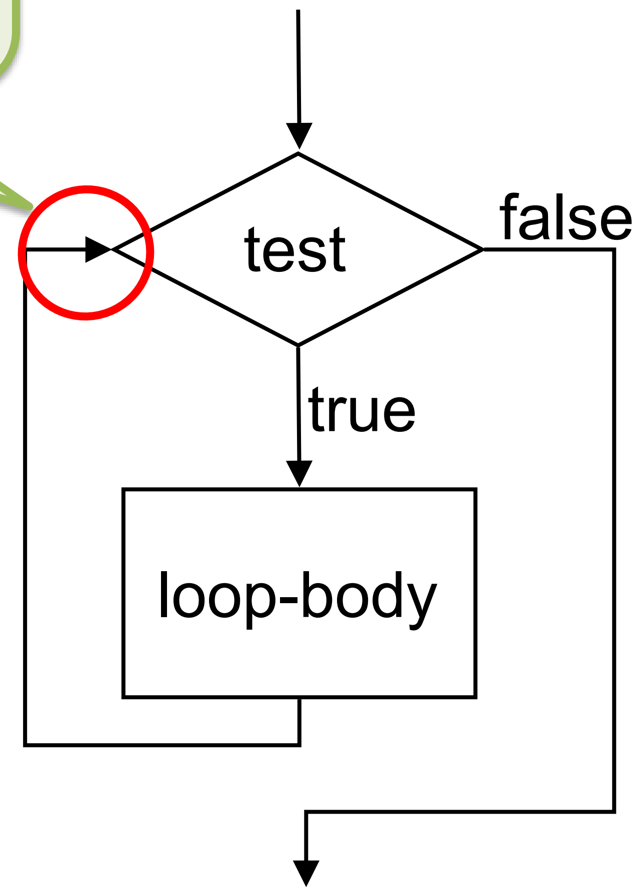
```
while (test) {
  loop-body
}
```



Control Flow

... and here, just after *every* execution of the loop body.

```
while (test) {
  loop-body
}
```



Example #1

`void append(Queue<T> q)`

- Concatenates ("appends") `q` to the end of **`this`**.
- Updates: **`this`**
- Clears: `q`
- Ensures:

  this *= #**this** * #q*

Example #1: Method Body

```
while (q.length() > 0) {
  T x = q.dequeue();
  this.enqueue(x);
}
```

Example #1: Method Body

```
while (q.length() > 0) {
   T x = q.dequeue();
   this.enqueue(x
}
```

What is true every time we test the loop condition? Lots of things... such as?

Code	State
	this *= < 1, 2, 3 >* *q = < 4, 5, 6 >*
while (q.length() > 0) {	
T x = q.dequeue();	
this.enqueue(x);	
}	

	this = < 1, 2, 3 > *q* = < 4, 5, 6 >
`while (q.length() > 0) {`	
`T x = q.dequeue();`	
`this.enqueue(x);`	
`}`	

What is true the *first* time we test the loop condition? Lots of things... such as?

	this *= < 1, 2, 3 >* *q = < 4, 5, 6 >*
`while (q.length() > 0) {`	
	this *= < 1, 2, 3 >* *q = < 4, 5, 6 >*
`T x = q.dequeue();`	
	this *= < 1, 2, 3 >* *q = < 5, 6 >* *x = 4*
`this.enqueue(x);`	
	this *= < 1, 2, 3, 4 >* *q = < 5, 6 >* *x = 4*
`}`	

Code	State
	this = < 1, 2, 3 > *q* = < 4, 5, 6 >
`while (q.length() > 0) {`	
	this = < 1, 2, 3 >
`T x = q.dequeue();`	
`this.enqueue(x);`	
	this = < 1, 2, 3, 4 > *q* = < 5, 6 > *x* = 4
`}`	

What is true the *first* and *second* times we test the loop condition?
Fewer things... such as?

	this *= < 1, 2, 3 >* *q = < 4, 5, 6 >*
`while (q.length() > 0) {`	
	this *= < 1, 2, 3 >*
`T x = q.dequeue();`	
`this.enqueue(x);`	
	th *= < 1, 2, 3, 4 >* *q < 5, 6 >* *x = 4*
`}`	

The value of `x` is not involved in the loop invariant because *there is no* `x` when we first hit the loop!

	this *= < 1, 2, 3 >* *q = < 4, 5, 6 >*
`while (q.length() > 0) {`	
	this *= < 1, 2, 3, 4 >* *q = < 5, 6 >*
`   T x = q.dequeue();`	
	this *= < 1, 2, 3, 4 >* *q = < 6 >* *x = 5*
`   this.enqueue(x);`	
	this *= < 1, 2, 3, 4, 5 >* *q = < 6 >* *x = 5*
`}`	

Code	State
	this = < 1, 2, 3 > *q* = < 4, 5, 6 >
`while (q.length() > 0) {`	
	this = < 1, 2, 3, 4 >
`T x = q.dequeue();`	
`this.enqueue(x);`	
	this = < 1, 2, 3, 4, 5 > *q* = < 6 > *x* = 5
`}`	

What is true the *first*, *second*, *and third* times we test the loop condition?
Fewer things still... such as?

	this = < 1, 2, 3 > *q* = < 4, 5, 6 >
`while (q.length() > 0) {`	
	this = < 1, 2, 3, 4, 5 > *q* = < 6 >
`T x = q.dequeue();`	
	this = < 1, 2, 3, 4, 5 > *q* = < > *x* = 6
`this.enqueue(x);`	
	this = < 1, 2, 3, 4, 5, 6 > *q* = < > *x* = 6
`}`	

	`this = < 1, 2, 3 >` `q = < 4, 5, 6 >`
`while (q.length() > 0) {`	
	`this = < 1, 2, 3, 4, 5 >`
`T x = q.dequeue();`	
`this.enqueue(x);`	

	`this = < 1, 2, 3, 4, 5, 6 >`
	`q = < >`
	`x = 6`

`}`	

What is true the *first*, *second*, *third*, *and fourth* times we test the loop condition?
Fewer things still... such as?

	this = < 1, 2, 3 > *q* = < 4, 5, 6 >
`while (q.length() > 0) {`	
	this = < 1, 2, 3, 4, 5 > *x* =
`T x = q.dequeue();`	
`this.enqueue(x);`	
	this = < 1, 2, 3, … , 6 > *q* = < > *x* = 6
`}`	

Whatever is true the last time we test the loop condition is also true here, after the loop finally terminates.

Some Things That Do Not Change

- "The lengths of the strings are non-negative" does not change
  - `|this| >= 0 and |q| >= 0` does not change
  - True, but this literally goes without saying; the length of any string is always non-negative
  - It is no more useful than saying, e.g., "`17 < 42` does not change", because it is a mathematical fact, not something about this loop in particular

Some Things That Do Not Change

- "The sum of the lengths of the strings" does not change
  - $|$***this***$| + |q|$ does not change
  - True, and a useful observation about this particular loop; but one can say more

Some Things That Do Not Change

- "The concatentation of the strings" does not change
  - `this * q` does not change
  - True, and a ***stronger*** useful observation about this loop because it *implies* the previous observation about the sum of the lengths
    - In other words, if `this * q` does not change, then `|this| + |q|` also does not change; but not vice versa

How To Express an Invariant

- How do we say “the concatenation of the strings does not change”?
  - We need to talk about both:
    - The ***current values*** of the variables
    - The ***original values*** of the variables, just before the loop condition was *first* tested (variable names prefixed with #)

`this * q = #this * #q`

Example #1: Method Body

```
/**
 * @updates this, q
 * @maintains
 * this * q = #this * #q
 */
while (q.length() > 0) {
  T x = q.dequeue();
  this.enqueue(x);
}
```

Example #1: Method Body

```
/**
 * @updates this, q
 * @maintains
 * this * q = #this * #q
 */
while (q.length() > 0)
  T x = q.dequeue();
  this.enqueue(x);
}
```

This Javadoc tag introduces the list of variables whose values might change in some iteration.

Example #1: Method Body

```
/**
 * @updates this, q
 * @maintains
 * this * q = #this * #q
 */
while (q.length() >
  T x = q.dequeue();
  this.enqueue(x);
}
```

Any variable in scope that is not listed as an ***updates-mode*** variable is, by default, a ***restores-mode*** variable, meaning the loop body does not change its value.

Example #1: Method Body

```
/**
 * @updates this, q
 * @maintains
 * this * q = #this * #q
 */
while (q.length() > 0)
  T x = q.dequeue();
  this.enqueue(x);
}
```

This Javadoc tag introduces the claim that the following loop “maintains” the property, i.e., it is a loop invariant.

Example #1: Method Body

```
/**
 * @updates this, q
 * @maintains
 * this * q = #this * #q
 */
while (q.length() > 0) {
  T x = q.dequeue();
  this.enqueue(x);
}
```

Using a Loop Invariant

- If you have a strong enough loop invariant, you can ***trace over*** a loop in a single step, and predict the values of the variables when it terminates—without tracing through the loop body even once

Example #1: Method Body

```
/**
 * @updates this, q
 * @maintains
 * this * q = #this * #q
 */
while (q.length() > 0) {
   ...
}
```

Pretend you cannot see the loop body. Can you still trace over this loop?

	this *= < 1, 2, 3 >* *q = < 4, 5, 6 >*
/** * **@maintains** * ***this*** ** q = #**this** * #q* */ **while** (q.length() > 0) {	
...	
}	
	this *=* *q =*

When execution reaches this point, we know two things...

	this = < 1, 2, 3 > q = < 4, 5, 6 >
/** * @maintains * this * q = #this * #q */ while (q.length() > 0) {	
...	
}	
	this = q =

We know (1) the loop invariant is true, so:

this * q = #this * #q

	this = < 1, 2, 3 > q = < 4, 5, 6 >
/** * @maintains * this * q = #this * #q */ while (q.length() > 0) {	
...	
}	
	this = q =

We also know (2) the loop condition is false, so:

`|q| <= 0`

Code	State
	this = < 1, 2, 3 > *q* = < 4, 5, 6 >

`/**`
` * @maintains`
` *   this * q = #this * #q`
` */`
`while (q.length() > 0) {`	

`   ...`	
`}`	
	this = < 1, 2, 3, 4, 5, 6 > *q* = < >

Combining (1) and (2), the only values the variables can possibly have at this point are these.

Justification for (1)

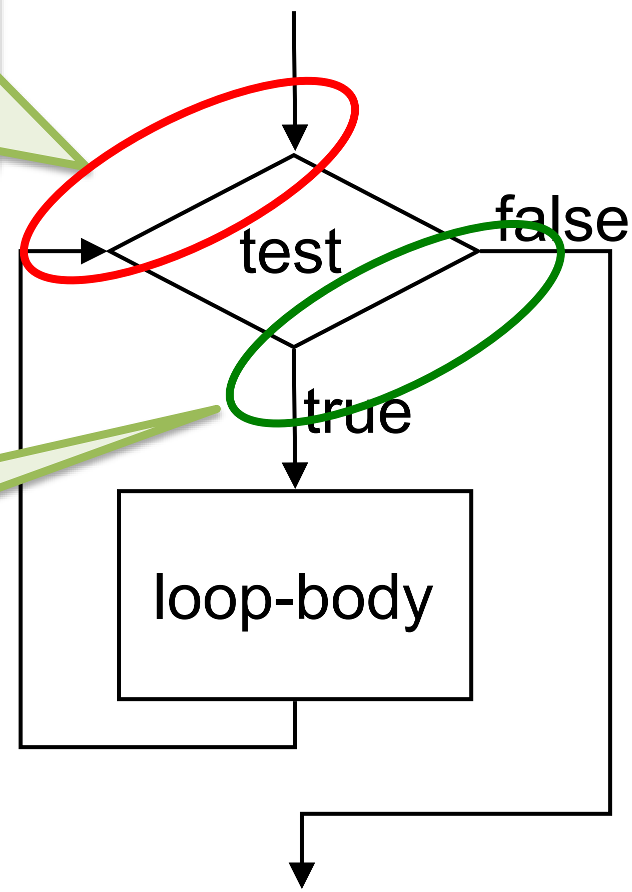
- The loop invariant is true just after the loop terminates—*if* the code that tests the loop condition does not change the value of any variable appearing in the loop invariant

The Loop Invariant Picture

If the loop invariant is true at the two red points, and “test” updates nothing...

```
while (test) {
    loop-body
}
```

...then the loop invariant is true at the two green points.



Justification for (1)

- **Best practice**: Code that tests the loop condition should not update any variables appearing in the loop invariant
  - Easy way to achieve this: the test should not update *any variables at all*

Justification for (2)

- The loop does not terminate until and unless the loop condition is false
  - However, a loop *might never terminate*; so you need to show that it does
  - This is similar to how you show a recursive method terminates

Loop Termination

- To show that a loop terminates, it is sufficient to provide a ***progress metric*** (a.k.a. ***termination function***, a.k.a. ***variant function***)
  - An integer-valued function of the variables in scope (where the loop appears in the code)
  - Always non-negative
  - Always decreases when the loop body is executed once

Example #1: Method Body

```
/**
 * @updates this, q
 * @maintains
 * this * q = #this * #q
 * @decreases
 * |q|
 */
while (q.length() > 0) {
  T x = q.dequeue();
  this.enqueue(x);
}
```

Example #1: Method Body

```
/**
 * @updates this, q
 * @maintains
 * this * q = #this * #q
 * @decreases
 * |q|
 */
while (q.length() > 0) {
   T x = q.dequeue();
   this.enqueue(x);
}
```

This Javadoc annotation claims the loop that follows “decreases” the stated progress metric.

Conclusion

- Even if you do not choose to *write down* a loop invariant or progress metric, if you *think about* loops in these terms it can help you avoid errors and bad practices in loop code
  - Off-by-one errors
  - Wrong/missing code in the loop body
  - Declarations of variables outside the loop that are only used inside the loop body