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On Connectedness: A Solution Based on Oscillatory Correlation

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A long-standing problem in neural computation has been the problem of connectedness, first identified by Minsky and Papert (1969). This problem served as the cornerstone for them to establish analytically that perceptrons are fundamentally limited in computing geometrical (topological) properties. A solution to this problem is offered by a different class of neural networks: oscillator networks. To solve the problem, the representation of oscillatory correlation is employed, whereby one pattern is represented as a synchronized block of oscillators and different patterns are represented by distinct blocks that desynchronize from each other. Oscillatory correlation emerges from LEGION (locally excitatory globally inhibitory oscillator network), whose architecture consists of local excitation and global inhibition among neural oscillators. It is further shown that these oscillator networks exhibit sensitivity to topological structure, which may lay a neurocomputational foundation for explaining the psychophysical phenomenon of topological perception.

Thirty years ago Minsky and Papert (1969), in their milestone book, pointed out fundamental limitations of perceptrons. They proved that all except one topologically invariant predicates are not of finite order. In particular, the topological predicate of connectedness was used as the cornerstone in their mathematical analysis. To understand the implications of this famous (negative) result, let us first explain what it means.

A perceptron (Rosenblatt, 1962) may be viewed as a binary classification device that computes a predicate. As shown in Figure 1, the class of perceptrons that Minsky and Papert studied consists of a binary input layer R (symbolizing retina), a layer of binary feature-detecting units, and a response unit that signifies the classification outcome. The feature detectors sense a specific area of R, and the response unit is a threshold unit operating on a weighted sum of all the units in the detector layer. The order of a predicate is the smallest number k for which one can compute the predicate with feature detectors that sense no more than k units of R. The Minsky-Papert connectedness theorem states that to tell whether an input pattern is connected requires arbitrarily large orders as R grows in size. In fact, the order increases at least as fast as $|R|^{1/2}$. Note that for any fixed R, the theo-

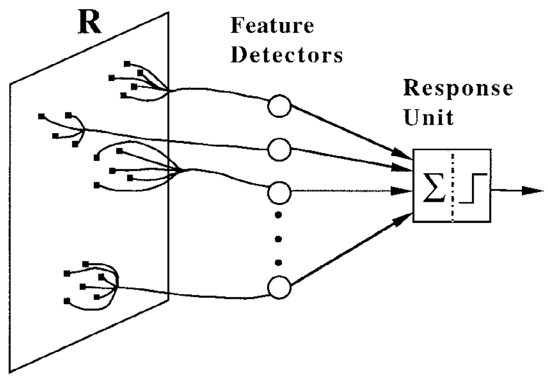

Figure 1: Diagram of a perceptron. A feature detector senses a specific area of the input layer, *R*, and the response unit computes whether the weighted sum of all the detectors is above a certain threshold.

rem does not prevent a theoretical solution to the problem by a perceptron. With a discrete *R*, there are a finite (but exponentially high except for one-dimensional *R*)[1] number of connected patterns, and one can trivially find a perceptron whose feature detectors detect individual connected patterns. However, problems that are not of finite order require nonlocal (global) feature detectors, and too many of them to be feasible (Minsky & Papert, 1969). In a sense, their result is a negative statement on the scalability of machinery.

Perceptrons are best known for their amazing ability to learn. What is powerful about the Minsky-Papert theory is that it identifies limitations on the kinds of patterns that perceptrons are capable of recognizing, regardless of whatever learning rule is employed. Much of the Minsky-Papert analysis is about single-layer (simple) perceptrons, and the following critical question arises: Is the limitation applicable to multilayer perceptrons with backpropagation learning, which represents the most influential development of modern neural networks? General analytical statements are not available. As far as connectedness is concerned, Minsky and Papert (1988, p. 52) claimed that multilayer perceptrons are no more powerful. For multilayer perceptrons, order is of less concern because hidden units in typical architecture receive input from all of the input units. The central questions are, How many hidden units are required? and How long is the training process? If the number or the duration increase very quickly as *R* grows in size, one must consider the problem not solvable in practice. For two-dimensional (2D) *R*, the number of connected patterns grows exponentially with respect to $|R|$. Thus, for *R* not too small, a practical training process

[1] I am grateful to N. Robertson for discussions that led to this conclusion.

can employ only a tiny percentage of all possible training samples, and it is hard to expect that a multilayer perceptron can successfully generalize from relatively so few training samples. Given this consideration and that no success has been reported on recognizing connectedness, it is reasonable to project, based on the result on simple perceptrons, that the limitation exists for multilayer perceptrons (see also Minsky & Papert, 1988).

Here we point out that a new class of neural networks—neural oscillator networks—provides a solution to the problem of connectedness. The basic unit of an oscillator network is a neural oscillator, and with an additional degree of freedom—the phase of an oscillator—one can speak of synchronization and desynchronization in an oscillator network. In particular, Terman and Wang (1995; Wang & Terman, 1997) proposed and analyzed LEGION (locally excitatory globally inhibitory oscillator network). Representation in these networks is based on oscillatory correlation, a special form of von der Malsburg's temporal correlation theory (von der Malsburg, 1981; von der Malsburg & Schneider, 1986). More specifically, a pattern is represented by a synchronized group of oscillators, and different patterns are represented by distinct oscillator groups that desynchronize from each other. A LEGION network consists of three parts: (1) its basic unit, a relaxation oscillator with two time scales; (2) oscillators coupled with local excitation, which leads to rapid synchronization within a group corresponding to one pattern; and (3) a global inhibitor, which desynchronizes different groups of oscillators.

Formally, a single oscillator i in LEGION is defined by a pair of excitatory variable x_i and inhibitory variable y_i. Oscillator i receives external stimulation I_i, and overall coupling S_i from the network. The definition of oscillator i includes a relaxation parameter ε, which induces two time scales that characterize typical relaxation oscillations. The limit cycle for a relaxation oscillation quickly alternates (or jumps) between an active phase of relatively high x values and a silent phase of relatively low x values. To study connectedness, we employ a 2D LEGION with four nearest-neighbor coupling and the global inhibitor z. (See Terman & Wang, 1995, and Wang & Terman, 1997, for precise definitions of LEGION.)

Given Terman-Wang analysis on LEGION, it is not difficult to use LEGION to detect connectedness. Before describing how to do it, we show the response of a 30×30 LEGION network to two figures: one connected and one disconnected. The connected figure is a cup shown in Figure 2A, while the disconnected one is the image of the word *cup* shown in Figure 2B. The LEGION network is solved using a Runge-Kutta method. The oscillators of the network start with random phases. Figure 2C displays the temporal activity of all the stimulated oscillators ($I_i > 0$) for the connected cup image. Unstimulated oscillators ($I_i < 0$) are omitted from the display because they do not oscillate. The oscillators corresponding to each pattern are combined in the display and thus appear as a single oscillator when they are in synchrony. The upper panel shows the oscillator block corresponding to the cup, and the middle panel shows the activity of the global inhibitor. Synchrony occurs in the first cycle of oscillations. The case for the disconnected *cup* is shown in Figure 2D, where the upper three traces show the three blocks corresponding to the three patterns, respectively. From Figure 2D, it is clear that synchronization within each block and desynchronization between different blocks are achieved in the first two cycles.

These simulations illustrate clearly that after a few cycles, all the oscillators corresponding to one (connected) pattern are synchronized, whereas those corresponding to different patterns are desynchronized. When a synchronized block jumps to the active phase, the global inhibitor is triggered and approaches from 0 to 1 on the fast timescale. Within an oscillation period, the global inhibitor is triggered as many times as is the number of patterns in the input figure. Thus, one can find how many patterns are in the input image by comparing the response of any stimulated oscillator and that of the global inhibitor. If their frequencies of oscillations are the same, then the input figure contains one pattern, and thus the figure is connected. Otherwise the input figure contains more than one pattern and the figure is disconnected. More specifically, we compute the accumulated activity of the global inhibitor over an oscillation period τ by $\int_{T-\tau}^{T} z\,dt$, where T denotes the current time. The corresponding average accumulated activity of a stimulated oscillator is given by

$$\frac{\sum_i \int_{T-\tau}^{T} H(x_i - \theta_x)\,dt}{\sum_i H(I_i)}.$$

Here H stands for the Heaviside step function, and θ_x is chosen between x values that correspond to the silent phase and those to the active phase. Note that the denominator indicates how many oscillators are stimulated.

Regarding the period τ of a stimulated oscillator, in the singular limit $\varepsilon \to 0$, it can be precisely expressed in terms of the parameters of a single

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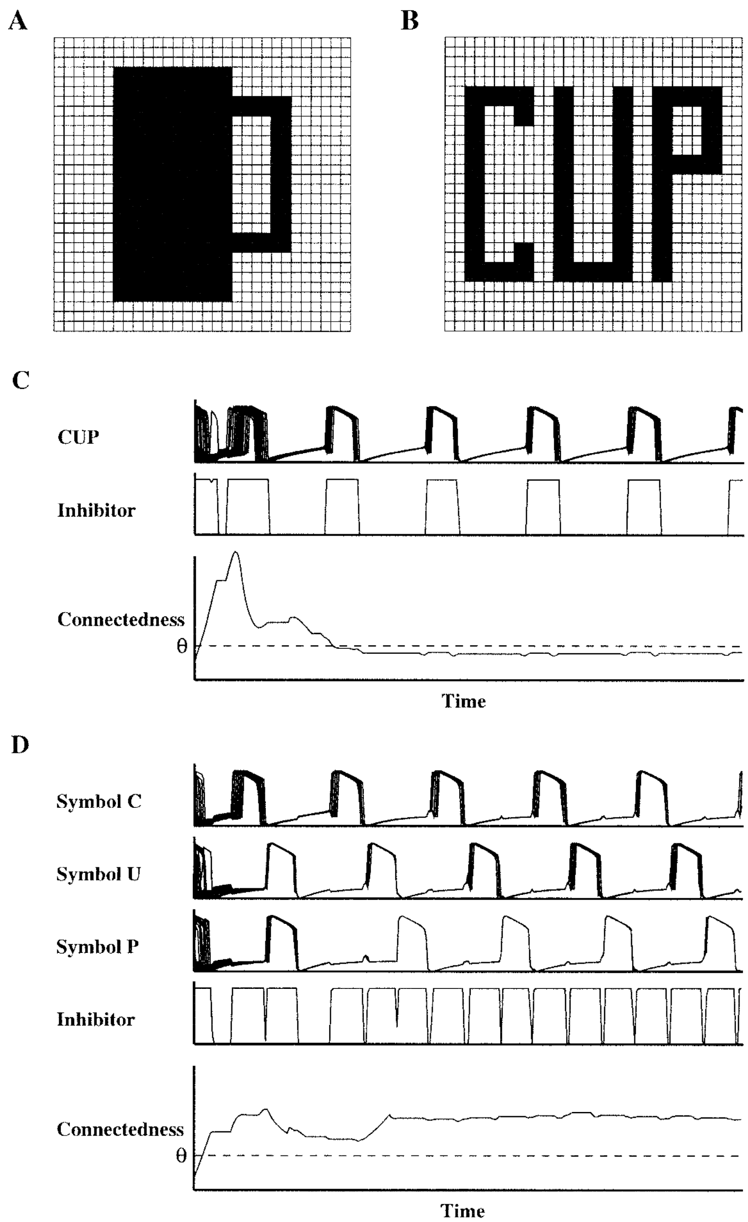
Figure 2: *Facing page*. (A) A connected cup image as mapped to a 30×30 LEGION. (B) A disconnected image with three patterns forming the word *cup* as mapped to a 30×30 network. (C) Results for the connected *cup* image. (D) Results for the disconnected *cup* image. In *C* and *D*, the upper traces give the combined x activities of the oscillator blocks indicated by their respective labels. The next-to-bottom trace gives the activity of the global inhibitor, and the bottom one shows the temporal activity of the RHS of equation 1, together with θ. The oscillator activity is normalized in the figure. The simulation in each case takes 7500 integration steps. The parameter values are (see Wang & Terman, 1997, for the meanings of these parameters): $\varepsilon = 0.02$, $\beta = 0.1$, $\gamma = 6.0$, $\rho = 0.02$, $\theta_x = -0.5$, $\theta_z = 0.1$, $\phi = 3.0$, $W_z = 1.0$, and $W_T = 8.0$ (weights are identical before dynamic normalization). $I = 0.2$ for a stimulated oscillator and $I = -0.02$ otherwise.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oscillator (see equation 10 in Linsay & Wang, 1998). When $\varepsilon > 0$ as in the previous simulations, the calculated period is not precise. But as will be clear later, we do not need precise values of τ in order to detect connectedness.

To summarize, the connectedness predicate is calculated by

$$\int_{T-\tau}^{T} z\,dt \Bigg/ \left(\frac{\sum_i \int_{T-\tau}^{T} H(x_i - \theta_x)\,dt}{\sum_i H(I_i)} \right) < \theta. \tag{1}$$

The threshold θ should be chosen between 1 and 2 for the following reason. Given a positive ε and system noise, synchrony within each pattern is not perfect (see Figure 2). Since z is triggered as long as a single oscillator jumps to the active phase, the width of the active phase of the inhibitor is slightly wider than a stimulated oscillator. Thus, for a connected figure, the right-hand side (RHS) of equation 1 is slightly greater than 1 but certainly less than 2.

The two bottom traces in Figures 2C and 2D show the RHS value of equation 1 for these two cases. For the parameter values used in the simulations, $\tau = 5.27$. A threshold $\theta = 1.6$ is used in Figure 2, and this is sufficient to detect connectedness. Beyond a short beginning duration corresponding to the process of synchronization and desynchronization, equation 1 correctly reveals the connectedness predicate.

More generally, the RHS of equation 1 reveals the number of connected components in the input figure. With this formula, the LEGION network can be used to count how many patterns are in a picture.

Note that equation 1 does not consider the degenerate case of no stimulus. What if there are numerous patterns in a picture? As analyzed in Wang and Terman (1997), for a fixed set of parameters, the LEGION network exhibits a fixed capacity in segmentation. If the number of patterns in an input picture exceeds the capacity, the network separates the picture into as many segments as the capacity. This observation, together with the result on the speed of synchronization and desynchronization (Terman & Wang, 1995; Wang & Terman, 1997), suggests that no matter how numerous patterns are in the input, one needs to wait for at most as many cycles as the capacity before connectedness can be correctly detected. With any capacity greater than 1, the connectedness predicate of equation 1 is not affected when numerous patterns appear in the input, because it is an assertion of whether the figure contains just one pattern. Thus, we obtain an upper bound on how long the system takes to compute the connectedness predicate. We emphasize that this detection of connectedness is analytically established, regardless of specific shapes, sizes, or arrangements of patterns in a picture.

Connectedness is not only computationally interesting, but also of fundamental significance to perception. It is one of the basic perceptual grouping principles (Rock & Palmer, 1990). According to Palmer and Rock (1994),

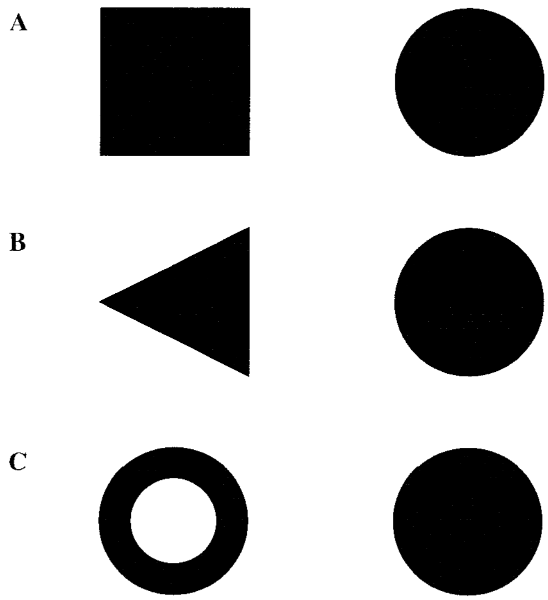

Figure 3: Three pairs of stimuli used in Chen's experiments (adapted from Chen, 1982).

connectedness accounts for the first-level organization. Furthermore, based on a series of psychophysical experiments Chen (1982, 1989) observed that the human visual system is sensitive to topological properties of stimuli, and topological perception constitutes a basic and early part of perceptual organization. Figure 3 illustrates a typical experiment with three pairs of stimuli, which manifest topological perception. Chen found that humans are more accurate in discriminating the annulus-disk pair under rapid display conditions. Note that the pairs in Figures 3A and 3B are topologically equivalent, whereas the annulus-disk pair is topologically distinct. Based on the difficulty of neural networks on connectedness, he challenged that topological perception remains a mystery for current neural network models (Chen, 1989).

Given that all six patterns in Figure 3 are connected, connectedness alone cannot explain the experimental result. However, we observe that although every pattern has an open "outside," only the annulus has an inside hole. With this observation, we can assume that a hole is a perceptually distinct pattern. Thus, the annulus consists of two connected patterns (or three, if one considers the open outside as a separate region), whereas the other five stimuli contain one pattern. When the stimuli in Figure 3 are presented to a LEGION network, all of them result in one segment, except for the annulus that results in two segments.[2] The qualitative difference in the number of segmented patterns emerging from LEGION provides a foundation for explaining topological perception. Our explanation applies equally well to cases involving more than one hole (Chen, 1990).

The fact that LEGION exhibits a fixed segmentation capacity results in the following novel and testable prediction: topology-based discrimination occurs only up to a certain number of holes. This prediction differs from that of Chen's (1990) topological hypothesis, rooted in mathematical topology.

To conclude, we have found a solution to the problem of detecting connectedness. The solution is given by a class of neural oscillator networks, which differ fundamentally from layered perceptrons. These networks are a kind of recurrent neural network and a dynamical system due to their continuous-time definitions. As such, oscillators in a network are effectively of unbounded order, which provides a necessary means to address the problem of connectedness. Furthermore, with their ability to exhibit sensitivity to topological structure, these oscillator networks provide a foundation for explaining psychophysical data of topological perception, and therefore have answered another challenge to neural networks. With their success in solving these difficult problems as well as their unique capability in scene analysis (Wang & Terman, 1997) and biological plausibility (Singer & Gray, 1995), oscillator networks may suggest a promising direction for future research in neural computation.

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[2] For simulations involving holes, as in Figure 3C, oscillators corresponding to holes are also stimulated and excitatory connections are formed only between neighbors with the same pixel levels (black or white).

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